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On condition (G-PWP)

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Abstract. Laan introduced the principal weak form of Condition (P) as Condition (PWP) and gave some characterization of monoids by this condition of their acts. In this paper first we introduce Condition (G-PWP), a generalization of Condition (PWP) of acts over monoids and then will give a characterization of monoids when all right acts satisfy this condition. We also give a characterization of monoids, by comparing this property of their acts with some others. Finally, we give a characterization of monoids coming from some special classes, by this property of their diagonal acts and extend some results on Condition (PWP) to this condition of acts.

1 Introduction

In [12], the concept of strong flatness was introduced: a right act A_S is strongly flat if the functor $A_S \otimes -$ preserves pullbacks and equalizers. In that article strongly flat acts were characterized as those acts that satisfy two interpolation conditions, later labelled Condition (P) and Condition (E) in [13]. In [10] Valdis Laan introduced the principal weak form of Condition (P) as Condition (PWP) and gave some characterization of monoids, by this condition of their acts.

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In this article in Section 2 first of all we introduce a generalization of Condition (PWP), called Condition (G-PWP) and will give some general properties. Then for a monoid S we will give a necessary and sufficient condition for a right S-act to satisfy this condition. We show that Condition (PWP) implies Condition (G -PWP), but not the converse, and Condition (G -PWP) implies GP-flatness, but the converse is not true in general. Then, we will give a characterization of monoids S over which all right S-acts satisfy Condition (G -PWP) and also a characterization of monoids S for which this condition of right S-acts has some other properties and vice versa. Some results from Condition (PWP) will also be extended to this property. Finally, in Section 3 we give a characterization of monoids coming from some special classes, by this property of their diagonal acts.

Throughout this article, $\mathbb{N}$ will stand for natural numbers. We refer the reader to [5] and [8] for basic definitions and results relating to acts over monoids and to [10] and [11] for definitions and results on flatness which are used here.

We use the following abbreviations,

weak pullback flatness = WPF.

weak kernel flatness = WKF.

principal weak kernel flatness = PWKF.

translation kernel flatness = TKF.

2 Characterization by condition (G-PWP) on right S-acts

We recall from [10] that a right S-act A_S satisfies *Condition* (PWP) if $as = a's$, for $a, a' \in A_S$ and $s \in S$, implies that there exist $a'' \in A_S$ and $u, v \in S$, such that $a = a''u$, $a' = a''v$ and $us = vs$.

Definition 2.1. Let S be a monoid and A_S a right S-act. We say that A_S satisfies *Condition* (G-PWP) if $as = a's$ for $a, a' \in A_S$ and $s \in S$, implies that there exist $a'' \in A_S$ and $u, v \in S$, $n \in \mathbb{N}$, such that $a = a''u$, $a' = a''v$ and $us^n = vs^n$.

Clearly, Condition (PWP) implies Condition (G-PWP), but not the converse, see the following example.

First we recall from [8] that a right ideal K of a monoid S is called *left stabilizing* if for every $k \in K$, there exists $l \in K$ such that $lk = k$. We also recall from [10] that K is called *left annihilating* if for all $s \in S$ and $x, y \in S \setminus K$, $xs, ys \in K$ implies that $xs = ys$.

Example 2.2. Let $S = \{1, 0, e, f, a\}$ be a monoid with the following table:

	1	0	e	f	a
1	1	0	e	f	a
0	0	0	0	0	0
e	e	0	e	a	a
f	f	0	0	f	0
a	a	0	0	a	0

If $K = aS = \{0, a\}$, then it is easy to see that the right Rees factor S-act S/K satisfies Condition (G-PWP). But K is not left annihilating, because, $a \in S$, $e, f \in S \setminus K$, $ea, fa \in K$ and $ea \neq fa$, also K is not left stabilizing, thus, by [8, III, 10.11], S/K is not principally weakly flat and so it does not satisfy Condition (PWP).

All statements in Proposition 2.3 are easy consequences of definition.

Proposition 2.3. *Let S be a monoid and A_S be a right S-act. Then*

(1) *S_S satisfies Condition (G-PWP).*

(2) *Θ_S satisfies Condition (G-PWP).*

(3) *Any retract of an act satisfying Condition (G-PWP) satisfies Condition (G-PWP).*

(4) *Let $A_S = \prod_{i \in I} A_i$, where A_i, $i \in I$, are right S-acts. If A_S satisfies Condition (G-PWP), then A_i satisfies Condition (G-PWP), for every $i \in I$.*

(5) *Let* $A_S = \coprod_{i\in I} A_i$*, where* A_i*,* $i \in I$*, are right* S*-acts. Then* A_S *satisfies Condition* (G-PWP) *if and only if each* A_i*,* $i \in I$*, satisfies Condition* (G-PWP)*.*

(6) *Let* $\{B_i | i \in I\}$ *be a chain of subacts of* A_S*. If every* B_i*,* $i \in I$*, satisfies Condition* (G-PWP)*, then* $\bigcup_{i\in I} B_i$ *satisfies Condition* (G-PWP)*.*

Proposition 2.4. *A right* S*-act* A_S *satisfies Condition* (G-PWP) *if and only if for all* $a, a' \in A_S$ *and all homomorphisms* $f : {}_SS \longrightarrow {}_SS$*, the equality* $af(s) = a'f(s)$ *for all* $s \in S$ *implies that there exist* $a'' \in A_S$*,* $u, v \in S$ *and* $n \in \mathbb{N}$ *such that* $a \otimes s = a'' \otimes u$*,* $a' \otimes s = a'' \otimes v$ *in* $A_S \otimes {}_SS$ *and* $uf^n(1) = vf^n(1)$*.*

Proof. Necessity. Suppose that A_S satisfies Condition (G-PWP) and let $af(s) = a'f(s)$, for homomorphism $f : {}_SS \longrightarrow {}_SS$, $a, a' \in A_S$ and $s \in S$. Then, $asf(1) = a'sf(1)$ and so there exist $a'' \in A_S$, $u, v \in S$ and $n \in \mathbb{N}$ such that $as = a''u$, $a's = a''v$ and $uf^n(1) = vf^n(1)$. Thus, by [8, II, 5.13], $a \otimes s = a'' \otimes u$ and $a' \otimes s = a'' \otimes v$ in $A_S \otimes {}_SS$, as required.
Sufficiency. Suppose that $as = a's$, for $a, a' \in A_S$, $s \in S$ and let $f : {}_SS \longrightarrow {}_SS$ be defined as $f(r) = rs$, $r \in S$. It is obvious that f is a homomorphism where $af(1) = a'f(1)$. Then, by assumption, there exist $a'' \in A_S$, $u, v \in S$ and $n \in \mathbb{N}$ such that $a \otimes 1 = a'' \otimes u$, $a' \otimes 1 = a'' \otimes v$ in $A_S \otimes {}_SS$ and $uf^n(1) = vf^n(1)$. Thus $us^n = vs^n$ and, by [8, II, 5.13], $a = a''u$, $a' = a''v$. Hence A_S satisfies Condition (G-PWP), as required. □

We recall from [7] that a right S-act A_S is called *GP-flat* if $a \otimes s = a' \otimes s$ in $A_S \otimes {}_SS$, for $a, a' \in A_S$, $s \in S$ implies that there exists $n \in \mathbb{N}$ such that $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes {}_SSs^n$.

Proposition 2.5. *Let* S *be a monoid and* A_S *be a right* S*-act. If* A_S *satisfies Condition* (G-PWP)*, then* A_S *is GP-flat.*

Proof. Suppose that A_S satisfies Condition (G-PWP) and let $as = a's$ for $a, a' \in A_S$ and $s \in S$. Then there exist $a'' \in A_S$, $u, v \in S$ and $n \in \mathbb{N}$ such that $a = a''u$, $a' = a''v$ and $us^n = vs^n$. Therefore,

$$a \otimes s^n = a''u \otimes s^n = a'' \otimes us^n = a'' \otimes vs^n = a''v \otimes s^n = a' \otimes s^n$$

in $A_S \otimes {}_SSs^n$, and so A_S is GP-flat, as required. □

The converse of Proposition 2.5 is not true, see the following example.

Example 2.6. Let $S = \{1, e, f, 0\}$ be a semilattice, where $ef = 0$. Consider the right ideal $K = eS = \{e, 0\}$ of S. Since K is left stabilizing, S/K is principally weakly flat, by [8, III, 10.11], and so it is GP-flat. But, it is easy to see that S-act S/K does not satisfy Condition (G-PWP).

We recall from [13] that a right S-act A_S satisfies *Condition* (E) if $as = at$, for $a \in A_S$ and $s, t \in S$, implies that there exist $a' \in A_S$ and $u \in S$, such that $a = a'u$ and $us = ut$. Also we recall from [9] that a right S-act A_S satisfies *Condition* (E') if $as = at$ and $sz = tz$, for $a \in A_S$ and $s, t, z \in S$, imply that there exist $a' \in A_S$ and $u \in S$, such that $a = a'u$ and $us = ut$. A right S-act A_S satisfies *Condition* (EP) if $as = at$ for $a \in A_S$ and $s, t \in S$, implies that there exist $a' \in A_S$ and $u, u' \in S$ such that $a = a'u = a'u'$ and $us = u't$. A right S-act A_S satisfies *Condition* $(E'P)$ if $as = at$ and $sz = tz$, for $a \in A_S$ and $s, t, z \in S$, imply that there exist $a' \in A_S$ and $u, u' \in S$ such that $a = a'u = a'u'$ and $us = u't$ (see [1], [2]).

It is obvious that $(E) \Rightarrow (E') \Rightarrow (E'P)$ and $(E) \Rightarrow (EP) \Rightarrow (E'P)$, but not the converses in general (see [1], [2]).

For monoids over which all right acts satisfy Condition (G-PWP), see the following proposition.

Proposition 2.7. *For any monoid S, the following statements are equivalent:*

(1) *all right S-acts satisfy Condition (G-PWP);*

(2) *all right S-acts satisfying Condition $(E'P)$ satisfy Condition (G-PWP);*

(3) *all right S-acts satisfying Condition (EP) satisfy Condition (G-PWP);*

(4) *all right S-acts satisfying Condition (E') satisfy Condition (G-PWP);*

(5) *all right S-acts satisfying Condition (E) satisfy Condition (G-PWP);*

(6) *all generators in* **Act**-*S satisfy Condition (G-PWP);*

(7) *$S \times A_S$ satisfies Condition (G-PWP), for every right S-act A_S;*

(8) *a right S-act A_S satisfies Condition (G-PWP) if $Hom(A_S, S_S) \neq \emptyset$;*

(9) *S is a group.*

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (5), (1) ⇒ (4) ⇒ (5), (9) ⇒ (1) and (1) ⇒ (6) are obvious.
(5) ⇒ (9). Suppose that I is a proper right ideal of S and let $A_S = S\coprod^I S$. Then

$$A_S = \{(\alpha, x)|\ \alpha \in S \setminus I\} \ \dot\cup\ I \ \dot\cup\ \{(\beta, y)|\ \beta \in S \setminus I\},$$

where $B_S = \{(\alpha, x)|\ \alpha \in S \setminus I\} \ \dot\cup\ I$ and $D_S = \{(\beta, x)|\ \beta \in S \setminus I\} \ \dot\cup\ I$ are subacts of A_S isomorphic to S_S. Since S_S satisfies Condition (E), B_S and D_S satisfy Condition (E), too, and so $A_S = B_S \cup D_S$ satisfies Condition (E) and so, by assumption, A_S satisfies Condition $(G\text{-}PWP)$. Hence, the equality $(1, x)t = (1, y)t$, for $t \in I$, implies that there exist $a \in A_S$, $u, v \in S$ and $n \in \mathbb{N}$ such that $(1, x) = au$, $(1, y) = av$ and $ut^n = vt^n$. Then equalities $(1, x) = au$ and $(1, y) = av$ imply, , that there exist $l, l' \in S \setminus I$ such that $a = (l, x)$ and $a = (l', y)$, which is a contradiction. Thus S has no proper right ideal, and so $aS = S$, for every $a \in S$. That is, S is a group, as required.
(6) ⇒ (7). It is obvious that the mapping $\pi : S \times A_S \to S_S$, where $\pi(s, a) = s$, for all $s \in S$ and $a \in A_S$, is an epimorphism in **Act**-S, and so $S \times A_S$ is a generator, by [8, II, 3.16], thus, by assumption, $S \times A_S$ satisfies Condition $(G\text{-}PWP)$.
(7) ⇒ (8). Suppose $Hom(A_S, S_S) \neq \emptyset$, for the right S-act A_S. We have to show that A_S satisfies Condition $(G\text{-}PWP)$. Let $f \in Hom(A_S, S_S)$, $as = a's$, for $a, a' \in A_S$ and $s \in S$. Then $f(as) = f(a's)$ and so $(f(a), a)s = (f(a'), a')s$ in $S \times A_S$. Thus there exist $(w, a'') \in S \times A_S$, $u, v \in S$ and $n \in \mathbb{N}$ such that $(f(a), a) = (w, a'')u$, $(f(a'), a') = (w, a'')v$ and $us^n = vs^n$. Therefore, $a = a''u$, $a' = a''v$ and $us^n = vs^n$, and so A_S satisfies Condition $(G\text{-}PWP)$, as required.
(8) ⇒ (1). Let A_S be a right S-act. It is obvious that the mapping π : $S \times A_S \to S_S$, where $\pi(s, a) = s$, for $s \in S$ and $a \in A_S$ is a homomorphism and so $Hom(S \times A_S, S_S) \neq \emptyset$. Let $as = a's$, for $a, a' \in A_S$ and $s \in S$. Then $(1, a)s = (1, a')s$ in $S \times A_S$, and so, by assumption, there exist $(w, a'') \in S \times A_S$, $u, v \in S$ and $n \in \mathbb{N}$ such that $(1, a) = (w, a'')u$, $(1, a') = (w, a'')v$ and $us^n = vs^n$. Then $a = a''u$, $a' = a''v$ and $us^n = vs^n$, and so A_S satisfies Condition $(G\text{-}PWP)$, as required. □

We recall from [8] that a right S-act A_S is *torsion free* if for $a, b \in A_S$ and a right cancellable element c of S, the equality $ac = bc$ implies that $a = b$. A_S is *strongly torsion free* if the equality $as = bs$ for all $a, b \in A_S$ and all $s \in S$ implies that $a = b$ (see [14]). Also we recall from [8] that an element $a \in A_S$ is called *act-regular* if there exists a homomorphism $f : aS \to S$ such that $af(a) = a$, and A_S is called a *regular act* if every $a \in A_S$ is an act-regular element.

An element $s \in S$ is called *generally left almost regular* if there exist elements $r, r_1, ..., r_m, s_1, ..., s_m \in S$, right cancellable elements $c_1, ..., c_m \in S$ and a natural number $n \in \mathbb{N}$ such that

$$s_1c_1 = sr_1$$

$$s_2c_2 = s_1r_2$$

$$\ldots$$

$$s_mc_m = s_{m-1}r_m$$

$$s^n = s_mrs^n.$$

A monoid S is called *generally left almost regular* if all its elements are generally left almost regular (see [7]).

An element $u \in S$ is called right *semi-cancellable* if for every $x, y \in S$, $xu = yu$ implies for some $r \in S$, $ru = u$ and $xr = yr$. A monoid S is *left PSF* if and only if every element of S is right semi-cancellative.

Definition 2.8. We say that a right ideal K of a monoid S is *G-left stabilizing* if for every $s \in S$ and $r \in S \setminus K$, $rs \in K$ implies that there exist $k \in K$ and $n \in \mathbb{N}$, such that $rs^n = ks^n$.

Proposition 2.5, [7, Proposition 2.6] and Example 2.6 show that Condition (*G-PWP*) of acts implies torsion freeness, but not the converse.

For the converse see the following proposition.

Proposition 2.9. *For any monoid S, the following statements are equivalent:*

(1) *all torsion free right S-acts satisfy Condition (G-PWP);*

(2) *all finitely generated torsion free right S-acts satisfy Condition $(G$-$PWP)$;*

(3) *all torsion free right S-acts generated by at most two elements satisfy Condition $(G$-$PWP)$;*

(4) *S is generally left almost regular and all GP-flat right S-acts satisfy Condition $(G$-$PWP)$;*

(5) *S is generally left almost regular and all finitely generated GP-flat right S-acts satisfy Condition $(G$-$PWP)$;*

(6) *S is generally left almost regular and all GP-flat right S-acts generated by at most two elements satisfy Condition $(G$-$PWP)$;*

(7) *S is left PSF and all GP-flat right S-acts satisfy Condition $(G$-$PWP)$;*

(8) *S is left PSF and all principally weakly flat right S-acts satisfy Condition $(G$-$PWP)$;*

(9) *S is left PSF and all weakly flat right S-acts satisfy Condition $(G$-$PWP)$;*

(10) *S is left PSF and all flat right S-acts satisfy Condition $(G$-$PWP)$;*

(11) *there exists a regular left S-act and all GP-flat right S-acts satisfy Condition $(G$-$PWP)$;*

(12) *there exists a regular left S-act and all principally weakly flat right S-acts satisfy Condition $(G$-$PWP)$;*

(13) *there exists a regular left S-act and all weakly flat right S-acts satisfy Condition $(G$-$PWP)$;*

(14) *there exists a regular left S-act and all flat right S-acts satisfy Condition $(G$-$PWP)$;*

(15) *there exists a regular left S-act and $|E(S)| = 1$;*

(16) *S is right cancellative.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (10) and (11) $\Rightarrow$ (12) $\Rightarrow$ (13) $\Rightarrow$ (14) are obvious.
(3) $\Rightarrow$ (6). Suppose that all torsion free right S-acts generated by at most two elements satisfy Condition (G-PWP). Since Condition (G-PWP) implies GP-flatness, all torsion free cyclic right S-acts are GP-flat and so S is generally left almost regular, by [7, Theorem 3.9]. Since GP-flatness implies torsion freeness, the second part is also true.
(1) $\Rightarrow$ (4). A similar argument as in (3) $\Rightarrow$ (6) can be used.
(16) $\Rightarrow$ (1). Suppose that S is a right cancellative monoid. Then all torsion free right S-acts are strongly torsion free, by [14, Corollary 3.1], and so we are done, because strong torsion freeness implies Condition (G-PWP).
(6) $\Rightarrow$ (16). Let C_r be the set of all right cancellable elements of S. If S is not right cancellative, then $C_r \neq S$. Let $I = S \setminus C_r$. Then $I \neq \emptyset$ and since $1 \in C_r$, $I \subset S$. Let $l \in I$ and $s \in S$, then there exist $l_1, l_2 \in S$ such that $l_1 \neq l_2$ and $l_1 l = l_2 l$, which implies that $l_1 ls = l_2 ls$. If $ls \in C_r = S \setminus I$, then the equality $l_1 ls = l_2 ls$ implies that $l_1 = l_2$, which is a contradiction. Thus $ls \in I = S \setminus C_r$, and so I is a right ideal of S. Now we show that I is G-left stabilizing. Let $rs \in I$, for $s \in S$ and $r \in S \setminus I = C_r$. Then $rs \in I$ implies that there exist $t_1, t_2 \in S$ such that $t_1 \neq t_2$ and $t_1 rs = t_2 rs$. By assumption, for $s \in S$, there exist elements $r^*, r_1, ..., r_m, s_1, ..., s_m \in S$, right cancellable elements $c_1, ..., c_m \in S$ and a natural number $n \in \mathbb{N}$ such that

$$s_1 c_1 = s r_1$$

$$s_2 c_2 = s_1 r_2$$

$$\ldots$$

$$s_m c_m = s_{m-1} r_m$$

$$s^n = s_m r^* s^n.$$

Since $t_1 rs = t_2 rs$, we have $t_1 rsr_1 = t_2 rsr_1$, using the first equality we have $t_1 rs_1 c_1 = t_2 rs_1 c_1$, and so $t_1 rs_1 = t_2 rs_1$.
Similarly, $t_1 rs_2 = t_2 rs_2, ..., t_1 rs_m = t_2 rs_m$. The last equality implies that $t_1 rs_m r^* = t_2 rs_m r^*$. If $s_m r^* = l$, then

$$t_1 rl = t_2 rl, \ ls^n = s_m r^* s^n = s^n \Rightarrow rs^n = (rl)s^n.$$

If $rl \in S \setminus I = C_r$, then the equality $t_1rl = t_2rl$ implies $t_1 = t_2$, which is a contradiction. Thus $rl \in I = S \setminus C_r$, and so $rs^n = (rl)s^n$ implies that $I = S \setminus C_r$ is G-left stabilizing. Thus the right S-act

$$A_S = S \coprod^{I} S = \{(\alpha, x) | \ \alpha \in S \setminus I\} \ \dot{\cup} \ I \ \dot{\cup} \ \{(\beta, y) | \ \beta \in S \setminus I\}$$

is GP-flat, by [7, Lemma 2.4], and so it satisfies Condition (G-PWP). Therefore the equality $(1, x)t = (1, y)t$, for $t \in I$ implies that there exist $a \in A_S$, $u, v \in S$ and $n \in \mathbb{N}$ such that $(1, x) = au$, $(1, y) = av$ and $ut^n = vt^n$. Then the equalities $(1, x) = au$ and $(1, y) = av$ imply, respectively, that there exist $l, l' \in S \setminus I$ such that $a = (l, x)$ and $a = (l', y)$, which is a contradiction. Thus S is a right cancellative monoid, as required.
(1) $\Rightarrow$ (7). It is true, because of (1) $\Leftrightarrow$ (16) and that every right cancellative monoid is left PSF.
(10) $\Rightarrow$ (16). Let S be a left PSF monoid, all flat right S-acts satisfy Condition (G-PWP), but S is not right cancellative. Let I be the set of all non cancellable elements of S. It is easy to see that I is a proper right ideal of S, where $i \in Ii$, for every $i \in I$. Then the right S-act

$$A_S = S \coprod^{I} S = \{(\alpha, x) | \ \alpha \in S \setminus I\} \ \dot{\cup} \ I \ \dot{\cup} \ \{(\beta, y) | \ \beta \in S \setminus I\}$$

is flat, by [8, III, 12.19]. Thus, by assumption, A_S satisfies Condition (G-PWP), which a similar argument as in the proof of (6) $\Rightarrow$ (16) shows that this is a contradiction. Thus S is a right cancellative monoid, as required.
(15) $\Leftrightarrow$ (16). It is true, by [6, Theorem 3.12].
(1) $\Rightarrow$ (11). It is true, since (1) $\Leftrightarrow$ (16) $\Leftrightarrow$ (15).
(14) $\Rightarrow$ (15). Suppose that there exist a regular left S-act, all flat right S-act satisfy Condition (G-PWP) and let $e \in E(S)$. If $eS = S$, then there exists $u \in S$ such that $eu = 1$, thus the equality $e(eu) = e$ implies that $e = 1$. If $eS \neq S$, then for every $i \in eS$ there exists $x \in S$ such that $i = ex$. Then $i = e(ex) = ei \in (eS)i$, and so the right S-act

$$S \coprod^{eS} S = \{(\alpha, x) | \ \alpha \in S \setminus eS\} \ \dot{\cup} \ eS \ \dot{\cup} \ \{(\beta, x) | \ \beta \in S \setminus eS\}$$

is flat, by [8, III, 12.19]. Thus, by assumption, it satisfies Condition (G-PWP), but a similar argument as in the proof of (6) $\Rightarrow$ (16) shows that this is a contradiction. Hence $E(S) = \{1\}$, as required. □

We recall from [8] that a right S-act A_S is *faithful* if for $s, t \in S$ the equality $as = at$, for all $a \in A$ implies that $s = t$, and A_S is *strongly faithful* if for $s, t \in S$ the equality $as = at$, for some $a \in A$ implies that $s = t$. It is obvious that every strongly faithful right $S-$act is faithful.

Lemma 2.10. *For any monoid S, the following statements are equivalent:*

(1) *there exists a strongly faithful cyclic right (left) S-act;*

(2) *there exists a strongly faithful finitely generated right (left) S-act;*

(3) *there exists a strongly faithful right (left) S-act;*

(4) *for every $s \in S$, sS (Ss) is a strongly faithful right (left) S-act;*

(5) *there exists $s \in S$ such that sS (Ss) is a strongly faithful right (left) S-act;*

(6) *S_S $({}_SS)$ is a strongly faithful right (left) S-act;*

(7) *for every $s \in S$, $sS \subseteq C_l$ $(Ss \subseteq C_r)$;*

(8) *there exists $s \in S$, $sS \subseteq C_l$ $(Ss \subseteq C_r)$;*

(9) *S is a left (right) cancellative monoid, that is, $S = C_l$ $(S = C_r)$ (C_l (C_r) is the set of all left (right) cancellable elements of S).*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (1), (9) $\Rightarrow$ (7) $\Rightarrow$ (8) and (6) $\Rightarrow$ (1) are obvious.
(3) $\Rightarrow$ (9). Suppose that A is a strongly faithful right (left) S-act, and let $sl = st$ $(ls = ts)$, for $l, t, s \in S$. Then for every $a \in A$, $asl = ast$ $(lsa = tsa)$. Since A is strongly faithful, the last equality implies that $l = t$. Hence S is a left (right) cancellative monoid, as required.
(9) $\Rightarrow$ (6). It is obvious.
(8) $\Rightarrow$ (9). Let $rt = rl$ $(tr = lr)$, for $l, t, r \in S$. Then $srt = srl$ $(trs = lrs)$ implies that $t = l$, and so S is a left (right) cancellative monoid, as required.
(9) $\Rightarrow$ (4). Suppose that S is a left (right) cancellative monoid and let $skt = skl$ $(tks = lks)$, for $l, k, t \in S$. Then $t = l$ and so sS (Ss) is a strongly faithful right (left) S-act, as required. □

Proposition 2.11. *For any monoid S, the following statements are equivalent:*

(1) *all strongly faithful right S-acts satisfy Condition $(G\text{-}PWP)$;*

(2) *all strongly faithful finitely generated right S-acts satisfy Condition $(G\text{-}PWP)$;*

(3) *all strongly faithful right S-acts generated by at most two elements satisfy Condition $(G\text{-}PWP)$;*

(4) *S is a group or S is not a left cancellative monoid.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.
(3) $\Rightarrow$ (4). If S is not left cancellative, then we are done. Otherwise, we suppose that there exists $s \in S$, such that $sS \neq S$. Then

$$A_S = S \coprod^{sS} S = \{(l,x) |\ l \in S \setminus sS\} \mathbin{\dot\cup} sS \mathbin{\dot\cup} \{(t,y) |\ t \in S \setminus sS\}$$

is a right S-act and $B_S = \{(l,x) |\ l \in S \setminus sS\} \mathbin{\dot\cup} sS \cong S \cong \{(t,y) |\ t \in S \setminus sS\} \mathbin{\dot\cup} sS = C_S$, such that $A_S = B_S \cup C_S$ is generated by two elements $(1,x)$ and $(1,y)$. Since S is left cancellative, it is strongly faithful, by Lemma 2.10, and so B_S and C_S are strongly faithful as subacts of A_S. Thus A_S is strongly faithful and so, by assumption, it satisfies Condition $(G\text{-}PWP)$. Thus the equality $(1,x)s = (1,y)s$, implies that there exist $a \in A_S$, $u, v \in S$ and $n \in \mathbb{N}$ such that $(1,x) = au$, $(1,y) = av$ and $us^n = vs^n$. Hence there exist $l, t \in S \setminus sS$ such that $a = (l,x) = (t,y)$, which is a contradiction. Thus $sS = S$, for every $s \in S$ and so S is a group, as required.
(4) $\Rightarrow$ (1). If S is not left cancellative, then we are done, by Lemma 2.10. Otherwise, by Proposition 2.7, it is obvious. □

Recall from [8] that a right S-act A_S is said to be *decomposable* if there exist two subacts $B_S, C_S \subseteq A_S$ such that $A_S = B_S \cup C_S$ and $B_S \cap C_S = \emptyset$. A right S-act which is not decomposable is called *indecomposable*.

S/K in Example 2.6 does not satisfy Condition $(G\text{-}PWP)$, but it is indecomposable. Thus indecomposablity does not imply Condition $(G\text{-}PWP)$ in general.

Also, let $S = (\mathbb{N}, .)$ and consider $A_S = \mathbb{N} \coprod^{\mathbb{N}\setminus\{1\}} \mathbb{N}$. Then $(1,x) \neq (1,y)$, but $(1,x)2 = 2 = (1,y)2$. Hence A_S is not torsion free and so does not satisfy Condition (*G-PWP*). But it can easily be seen that A_S is faithful. Thus faithfulness does not imply Condition (*G-PWP*) in general.

Now we give a characterization of monoids S for which indecomposablity or faithfulness of right S-acts implies Condition (*G-PWP*).

Proposition 2.12. *For any monoid S, the following statements are equivalent:*

(1) *all indecomposable right S-acts satisfy Condition (G-PWP);*

(2) *all indecomposable finitely generated right S-acts satisfy Condition (G-PWP);*

(3) *all indecomposable right S-acts generated by at most two elements satisfy Condition (G-PWP);*

(4) *all faithful right S-acts satisfy Condition (G-PWP);*

(5) *all faithful finitely generated right S-acts satisfy Condition (G-PWP);*

(6) *all faithful right S-acts generated by at most two elements satisfy Condition (G-PWP);*

(7) *S is a group.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (4) and (7) $\Rightarrow$ (1) are obvious.
(3) $\Rightarrow$ (7). Suppose that I is a proper right ideal of S. Since

$$A_S = S\coprod^{I} S = \{(\alpha, x)|\ \alpha \in S \setminus I\} \,\dot\cup\, I \,\dot\cup\, \{(\beta, x)|\ \beta \in S \setminus I\}$$

is an indecomposable right S-act generated by $(1, x)$ and $(1, y)$, it satisfies Condition (*G-PWP*), but a similar argument as in the proof of Proposition 2.7 shows that this is a contradiction. Thus S has no proper ideal, that is, S is a group, as required.
(6) $\Rightarrow$ (7). Suppose that I is a proper right ideal of S and let

$$A_S = S\coprod^{I} S = \{(\alpha, x)|\ \alpha \in S \setminus I\} \,\dot\cup\, I \,\dot\cup\, \{(\beta, x)|\ \beta \in S \setminus I\}.$$

Then for $s \neq t \in S$, there exists $(1,x) \in A_S$ such that $(1,x)s \neq (1,x)t$, that is, A_S is a faithful right S-act. Thus, by assumption, A_S satisfies Condition (G-PWP), but a similar argument as in the proof of Proposition 2.7 shows that this is a contradiction. Hence, S has no proper ideal, that is, S is a group, as required. □

For elements $u, v \in S$, the relation $P_{u,v}$ is defined on S as

$$(x,y) \in P_{u,v} \Leftrightarrow ux = vy (x, y \in S).$$

and Δ_S denotes the diagonal congruence, i.e. $\Delta_S = \{(s,s) | s \in S\}$.

Lemma 2.13. *Let S be a monoid. Then:*

(1) $(\forall s \in S) P_{1,s} \circ \ker \lambda_s \circ P_{s,1} = \Delta_S \cap (sS \times sS)$;

(2) $(\forall u, v, s \in S)(\forall n \in \mathbb{N})$

$$(P_{u,v} \subseteq P_{1,s} \circ \ker \lambda_s \circ P_{s,1} \;\wedge\; us^n = vs^n) \Longleftrightarrow$$

$$((s^n S \times s^n S) \cap \Delta_S \subseteq P_{u,v} \subseteq (sS \times sS) \cap \Delta_S));$$

Proof. (1). Let $l_1, l_2 \in S$. Then:
$((l_1, l_2) \in P_{1,s} \circ \ker \lambda_s \circ P_{s,1}) \Longleftrightarrow \Big((\exists y_1, y_2 \in S)(l_1, y_1) \in P_{1,s} \wedge (y_1, y_2) \in \ker \lambda_s \wedge (y_2, l_2) \in P_{s,1}) \Longleftrightarrow ((\exists y_1, y_2 \in S)\; l_1 = sy_1 \wedge sy_1 = sy_2 \wedge sy_2 = l_2) \Longleftrightarrow ((\exists y_1, y_2 \in S)\; l_1 = sy_1 = sy_2 = l_2) \Longleftrightarrow \Big((l_1, l_2) \in \Delta_S \cap (sS \times sS)),$
as required.
(2). First we suppose that $P_{u,v} \subseteq P_{1,s} \circ \ker \lambda_s \circ P_{s,1}$ and $us^n = vs^n$, for $u, v, s \in S$ and $n \in \mathbb{N}$, we show that:

$$(s^n S \times s^n S) \cap \Delta_S \subseteq P_{u,v} \subseteq (sS \times sS) \cap \Delta_S.$$

By (1), it is obvious that $P_{u,v} \subseteq (sS \times sS) \cap \Delta_S$. Now let $(l_1, l_2) \in (s^n S \times s^n S) \cap \Delta_S$. Then there exist $y_1, y_2 \in S$ such that $l_1 = s^n y_1 = s^n y_2 = l_2$. Thus the equality $us^n = vs^n$ implies that

$$ul_1 = us^n y_1 = us^n y_2 = vs^n y_2 = vl_2.$$

Thus $(l_1, l_2) \in P_{u,v}$, and so

$$(s^n S \times s^n S) \cap \Delta_S \subseteq P_{u,v} \subseteq (sS \times sS) \cap \Delta_S,$$

as required.
For the other side, using (1), we have $P_{u,v} \subseteq P_{1,s} \circ \ker \lambda_s \circ P_{s,1}$ and since $(s^n, s^n) \in (s^nS \times s^nS) \cap \Delta_S \subseteq P_{u,v}$, we have $us^n = vs^n$. □

Proposition 2.14. *For any monoid S, the following statements are equivalent:*

(1) *all fg-weakly injective right S-acts satisfy Condition (G-PWP);*

(2) *all weakly injective right S-acts satisfy Condition (G-PWP);*

(3) *all injective right S-acts satisfy Condition (G-PWP);*

(4) *all cofree right S-acts satisfy Condition (G-PWP);*

(5) $(\forall s \in S)(\exists u, v \in S)(\exists n \in \mathbb{N})$

$\ker \lambda_u = \ker \lambda_v = \Delta_S \;\wedge\; P_{u,v} \subseteq P_{1,s} \circ \ker \lambda_s \circ P_{s,1} \;\wedge\; us^n = vs^n$;

(6) $(\forall s \in S)(\exists u, v \in S)(\exists n \in \mathbb{N})$

$\ker \lambda_u = \ker \lambda_v = \Delta_S \;\wedge\; P_{1,s^n} \circ \ker \lambda_{s^n} \circ P_{s^n,1} \subseteq P_{u,v} \subseteq$

$P_{1,s} \circ \ker \lambda_s \circ P_{s,1}$;

(7) $(\forall s \in S)(\exists u, v \in S)(\exists n \in \mathbb{N})$

$\ker \lambda_u = \ker \lambda_v = \Delta_S \;\wedge\; (s^nS \times s^nS) \cap \Delta_S \subseteq P_{u,v} \subseteq$

$(sS \times sS) \cap \Delta_S$.

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious.
Implications (5) ⟺ (6) ⟺ (7) are true, by Lemma 2.13.
(4) ⇒ (5). Suppose that all cofree right S-acts satisfy Condition (G-PWP), S_1, S_2 are the sets, where $|S_1| = |S_2| = |S|$, and $\alpha : S \longrightarrow S_1$, $\beta : S \longrightarrow S_2$ are bijections.
Let $s \in S$, $X = S/\ker \lambda_s \,\dot{\cup}\, S_1 \,\dot{\cup}\, S_2$ and define the mappings $f, g : S \longrightarrow X$ as

$$f(x) = \begin{cases} [y]_{\ker \lambda_s} & \text{if there exists } y \in S; x = sy \\ \alpha(x) & \text{if } x \in S \setminus sS. \end{cases}$$

$$g(x) = \begin{cases} [y]_{\ker \lambda_s} & \text{if there exists } y \in S; x = sy \\ \beta(x) & \text{if } x \in S \setminus sS. \end{cases}$$

We show that f is well-defined. For this, we suppose that $sy_1 = sy_2$, for $y_1, y_2 \in S$, hence $(y_1, y_2) \in \ker \lambda_s$ and so $[y_1]_{\ker \lambda_s} = [y_2]_{\ker \lambda_s}$, that is, $f(sy_1) = f(sy_2)$ and so f is well-defined. Similarly, g is well-defined. Since $fs = gs$, and $X^S = \{h : S \longrightarrow X |\ h \text{ is mapping}\}$ satisfies Condition (G-PWP), there exist a mapping $h : S \longrightarrow X$, $u, v \in S$ and $n \in \mathbb{N}$, such that $f = hu$, $g = hv$ and $us^n = vs^n$. Let $(l_1, l_2) \in \ker \lambda_u$, for $l_1, l_2 \in S$, then

$$ul_1 = ul_2 \Rightarrow f(l_1) = (hu)(l_1) = h(ul_1) = h(ul_2) = (hu)l_2 = f(l_2) \Rightarrow$$

$$f(l_1) = f(l_2) \Rightarrow \quad l_1, l_2 \in sS \ \vee \ l_1, l_2 \in S \setminus sS$$

if $l_1, l_2 \in S \setminus sS$, then

$$\alpha(l_1) = f(l_1) = f(l_2) = \alpha(l_2) \ \Rightarrow \ l_1 = l_2.$$

If $l_1, l_2 \in sS$, then there exist $y_1, y_2 \in S$ such that $l_1 = sy_1$ and $l_2 = sy_2$, hence

$$f(l_1) = f(sy_1) = [y_1]_{\ker \lambda_s},\ f(l_2) = f(sy_2) = [y_2]_{\ker \lambda_s}$$

$$f(l_1) = f(l_2) \Rightarrow [y_1]_{\ker \lambda_s} = [y_2]_{\ker \lambda_s} \ \Rightarrow \ (y_1, y_2) \in \ker \lambda_s$$

$$sy_1 = sy_2 \ \Rightarrow \ l_1 = l_2$$

thus the equality $f(l_1) = f(l_2)$ implies that $l_1 = l_2$, and $\ker \lambda_u = \Delta_S$. Analogously, the equality $g = hv$ implies that $\ker \lambda_v = \Delta_S$. Suppose now that $(x, y) \in P_{u,v}$. Then $ux = vy$, and so

$$f(x) = (hu)(x) = h(ux) = h(vy) = (hv)y = g(y) \ \Rightarrow \ f(x) = g(y).$$

The last equality implies that $x, y \in sS$ and so there exist $t_1, t_2 \in S$ such that $x = st_1,\ y = st_2$, hence $f(x) = [t_1]_{\ker \lambda_s}$ and $g(y) = [t_2]_{\ker \lambda_s}$. Thus

$$f(x) = g(y) \ \Rightarrow \ [t_1]_{\ker \lambda_s} = [t_2]_{\ker \lambda_s} \ \Rightarrow \ (t_1, t_2) \in \ker \lambda_s,$$

and so we have

$$(x, t_1) \in P_{1,s} \ \wedge \ (t_1, t_2) \in \ker \lambda_s \wedge \ (t_2, y) \in P_{s,1}$$

$$\Rightarrow \ (x, y) \in P_{1,s} \circ \ker \lambda_s \circ P_{s,1} \ \Rightarrow \ P_{u,v} \subseteq P_{1,s} \circ \ker \lambda_s \circ P_{s,1}.$$

(7) $\Rightarrow$ (1). Suppose that A_S is an fg-weakly injective right S-act and let $as = a's$, for $a, a' \in A_S$ and $s \in S$. By assumption, there exist $u, v \in S$ and $n \in \mathbb{N}$, such that

$$\ker \lambda_u = \ker \lambda_v = \Delta_S, \ (s^n S \times s^n S) \cap \Delta_S \subseteq P_{u,v} \subseteq (sS \times sS) \cap \Delta_S.$$

Define the mapping $\varphi : uS \cup vS \longrightarrow A$, such that for every $x \in uS \cup vS$,

$$\varphi(x) = \begin{cases} ap & \text{if there exists } p \in S; \ x = up \\ a'q & \text{if there exists } p \in S; \ x = vq \end{cases}$$

First we show that φ is well-defined. If there exist $p_1, p_2 \in S$ such that $up_1 = up_2$, then

$$(p_1, p_2) \in \ker \lambda_u = \Delta_S \ \Rightarrow \ p_1 = p_2 \ \Rightarrow \ ap_1 = ap_2$$

If there exist $q_1, q_2 \in S$, such that $vq_1 = vq_2$, then

$$(q_1, q_2) \in \ker \lambda_v = \Delta_S \ \Rightarrow \ q_1 = q_2 \ \Rightarrow \ a'q_1 = a'q_2$$

If there exist $p, q \in S$ such that $up = vq$, then $(p, q) \in P_{u,v} \subseteq (sS \times sS) \cap \Delta_S$ and so there exist $l_1, l_2 \in S$ such that $p = sl_1 = sl_2 = q$, which implies that

$$ap = asl_1 = asl_2 = a'sl_2 = a'q.$$

Thus, φ is well-defined, and obviously it is a homomorphism. Since, by assumption, A_S is an fg-weakly injective right S-act, there exists an extension $\psi : S \longrightarrow A_S$ of φ. If $a'' = \psi(1)$, then $a = \varphi(u) = \psi(u) = \psi(1)u = a''u$ and $a' = \varphi(v) = \psi(v) = \psi(1)v = a''v$. Also, by assumption,

$$(s^n, s^n) \in (s^n S \times s^n S) \cap \Delta_S \subseteq P_{u,v} \ \Rightarrow \ us^n = vs^n,$$

hence A_S satisfies Condition (*G-PWP*), as required. □

Notice that in Proposition 2.14, $\ker \lambda_u = \ker \lambda_v = \Delta_S$ if and only if u and v is left cancellable.

Corollary 2.15. *Let S be a monoid such that the set of all left cancellable elements are commutative. Then all cofree right S-acts satisfy Condition (G-PWP) if and only if S is a group.*

Proof. Necessity. Suppose that all cofree right S-acts satisfy Condition (G-PWP). By Proposition 2.14, for every $s \in S$ there exist $u, v \in S$ and $n \in \mathbb{N}$ such that

$$\ker \lambda_u = \ker \lambda_v = \Delta_S \ \wedge \ (s^n S \times s^n S) \cap \Delta_S \subseteq P_{u,v} \subseteq (sS \times sS) \cap \Delta_S.$$

Thus u and v are left cancellable and so, by assumption, $uv = vu$. Hence,

$$(v, u) \in P_{u,v} \subseteq (sS \times sS) \cap \Delta_S \Rightarrow u = v$$

$$\Delta_S \subseteq \ker \lambda_u = P_{u,u} = P_{u,v} \subseteq (sS \times sS) \cap \Delta_S \subseteq \Delta_S$$
$$\Rightarrow \ker \lambda_u = \Delta_S = (sS \times sS) \cap \Delta_S \subseteq sS \times sS$$
$$\Rightarrow (1, 1) \in \Delta_S \subseteq sS \times sS \Rightarrow \exists x \in S, 1 = sx$$

Thus $sS = S$, and so S is a group, as required.
Sufficiency is true, by Proposition 2.7. □

Notice that, Corollary 2.15 holds for any monoid S with $C_l(S) \subseteq C(S)$ or $C(S) = S$ ($C(S)$ is the center of S).

Corollary 2.16. *Let S be a finite monoid. Then all cofree right S-acts satisfy Condition (G-PWP) if and only if S is a group.*

Proof. Necessity. By Proposition 2.14, for every $s \in S$ there exist $u, v \in S$ and $n \in \mathbb{N}$ such that

$$\ker \lambda_u = \ker \lambda_v = \Delta_S \ \wedge \ (s^n S \times s^n S) \cap \Delta_S \subseteq P_{u,v} \subseteq ((sS \times sS) \cap \Delta_S).$$

On the other hand

$$uS \cong S/ker\lambda_u = S/\Delta_S \cong S \Rightarrow uS \cong S \Rightarrow |uS| = |S|$$

Since $uS \subseteq S$ and S is finite we have $uS = S$. Thus there exists $x \in S$ such that $ux = v$, and so we have

$$(x, 1) \in P_{u,v} \subseteq (sS \times sS) \cap \Delta_S \Rightarrow x = 1 \Rightarrow u = v.$$

Now a similar argument as in the proof of Corollary 2.15 shows that $sS = S$. That is, S is a group, as required.
Sufficiency is obvious, by Proposition 2.7. □

Corollary 2.17. *Let S be a monoid and suppose every left cancellable element of S has a right inverse. Then all cofree right S-acts satisfy Condition $(G\text{-}PWP)$ if and only if S is a group.*

Proof. Since, by assumption, $uS = S$, for any $u \in C_l(S)$, a similar argument as in the proof of Corollary 2.16 can be used. □

Notice that, for finite monoids, every left cancellable element has a right inverse.

Corollary 2.18. *Let S be an idempotent monoid. Then all cofree right S-acts satisfy Condition $(G\text{-}PWP)$ if and only if $S = \{1\}$.*

Proof. Necessity. If $e \in S$, then, by Proposition 2.14, there exist $u, v \in S$ such that

$$\ker \lambda_u = \ker \lambda_v = \Delta_S,\ P_{u,v} = (eS \times eS) \cap \Delta_S.$$

Thus $(u, 1) \in \ker \lambda_u = \Delta_S$, which implies that $u = 1$, similarly $v = 1$. So we have

$$\Delta_S = \ker \lambda_1 = P_{u,v} = P_{u,u} = (eS \times eS) \cap \Delta_S \subseteq (eS \times eS)$$

Then $(1, 1) \in \Delta_S \subseteq (eS \times eS)$ implies that there exists $x \in S$ such that $ex = 1$, and so $e = 1$, that is, $S = \{1\}$, as required.
Sufficiency is clear. □

So far there is no characterization of monoids for which (fg-weak, weak) injectivity or cofreeness imply Condition (PWP). For a characterization of these monoids see the following corollary.

Corollary 2.19. *For any monoid S, the following statements are equivalent:*

(1) *all fg-weakly injective right S-acts satisfy Condition (PWP);*

(2) *all weakly injective right S-acts satisfy Condition (PWP);*

(3) *all injective right S-acts satisfy Condition (PWP);*

(4) *all cofree right S-acts satisfy Condition (PWP);*

(5) $(\forall s \in S)(\exists u, v \in S)$

$$(\ker \lambda_u = \ker \lambda_v = \Delta_S \;\wedge\; P_{u,v} = P_{1,s} \circ \ker \lambda_s \circ P_{s,1});$$

(6) $(\forall s \in S)(\exists u, v \in S)$

$$(\ker \lambda_u = \ker \lambda_v = \Delta_S \;\wedge\; P_{u,v} = (sS \times sS) \cap \Delta_S).$$

Proof. Apply Proposition 2.14, for $n = 1$. □

Recall from [8] that, a right S-act A_S satisfies *Condition* (P) if $as = a't$, for $a, a' \in A_S$, $s, t \in S$, there exist $a'' \in A_S$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vt$. Also we recall from [4] that a right S-act A_S satisfies *Condition* (P') if $as = a't$ and $sz = tz$, for $a, a' \in A_S$, $s, t, z \in S$, imply that there exist $a'' \in A_S$ and $u, v \in S$, such that $a = a''u$, $a' = a''v$ and $us = vt$.

We know that

$$WPF \Rightarrow WKF \Rightarrow PWKF \Rightarrow TKF \Rightarrow (PWP) \Rightarrow (G\text{-}PWP)$$
$$WPF \Rightarrow (P) \Rightarrow (WP) \Rightarrow (PWP) \Rightarrow (G\text{-}PWP)$$
$$(P) \Rightarrow (P') \Rightarrow (PWP) \Rightarrow (G\text{-}PWP).$$

Now, let (U) be a property of acts that can be stand for WPF, WKF, $PWKF$, TKF, (P), (WP), (P') or (PWP), then, by Corollaries 2.15, 2.16, 2.17 and [11, Proposition 9], we have the following corollary.

Corollary 2.20. *Let S be a monoid for which one of the following conditions is satisfied:*

(1) *$C_l(S)$ is commutative;*

(2) *S is finite;*

(3) *$cS = S$, for every $c \in C_l(S)$.*

Then all cofree right S-acts satisfy Condition (U) if and only if S is a group.

Corollary 2.21. *Let S be an idempotent monoid and let (U) be a property of acts that can be stand for free, projective generator, projective, strongly flat, WPF, WKF, PWKF, TKF, (P), (WP), (P') or (PWP). Then all cofree right S-acts satisfy Condition (U) if and only if $S = \{1\}$.*

Proof. By Corollary 2.18, it is obvious. □

By Proposition 2.3, S_S and Θ_S satisfy Condition (*G-PWP*) for any monoid S. But Θ_S is faithful if and only if $S = \{1\}$, and S_S is strongly faithful if and only if S is left cancellative. Thus Condition (*G-PWP*) of acts does not imply (strong) faithfulness in general. The following proposition gives a characterization of monoids S for which Condition (*G-PWP*) of right S-acts implies (strong) faithfulness.

Proposition 2.22. *For any monoid S, the following statements are equivalent:*

(1) *all right S-acts satisfying Condition* (*G-PWP*) *are* (*strongly*) *faithful;*

(2) *all finitely generated right S-acts satisfying Condition* (*G-PWP*) *are* (*strongly*) *faithful;*

(3) *all cyclic right S-acts satisfying Condition* (*G-PWP*) *are* (*strongly*) *faithful;*

(4) *all Rees factor right S-acts satisfying Condition* (*G-PWP*) *are* (*strongly*) *faithful;*

(5) $S = \{1\}$.

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) and (5) ⇒ (1) are obvious.
(4) ⇒ (5). Since $\Theta_S = S/S_S$ satisfies Condition (*G-PWP*), it is (strongly) faithful, and so $S = \{1\}$. □

Example 2.2, shows that Condition (*G-PWP*) of acts does not imply freeness and projective generator. For a characterization of monoids when this is the case see the following proposition.

Proposition 2.23. *For any monoid S, the following statements are equivalent:*

(1) *all right S-acts satisfying Condition* (*G-PWP*) *are free;*

(2) *all right S-acts satisfying Condition* (*G-PWP*) *are projective generators;*

(3) *all finitely generated right S-acts satisfying Condition (G-PWP) are free;*

(4) *all finitely generated right S-acts satisfying Condition (G-PWP) are projective generators;*

(5) *all cyclic right S-acts satisfying Condition (G-PWP) are free;*

(6) *all cyclic right S-acts satisfying Condition (G-PWP) are projective generators;*

(7) *all monocyclic right S-acts satisfying Condition (G-PWP) are free;*

(8) *all monocyclic right S-acts satisfying Condition (G-PWP) are projective generators;*

(9) $S = \{1\}$.

Proof. Implications (1) ⇒ (2) ⇒ (4) ⇒ (6) ⇒ (8), (1) ⇒ (3) ⇒ (5) ⇒ (7), (3) ⇒ (4), (5) ⇒ (6), (7) ⇒ (8) and (9) ⇒ (1) are obvious.
(8) ⇒ (9): By [8, IV, 12.8], it is obvious. □

We recall from [8] that an element $s \in S$ is called *left almost regular* if there exist $r, r_1, ..., r_m, s_1, s_2, ..., s_m \in S$ and right cancellable elements $c_1, c_2, ..., c_m \in S$ such that

$$s_1c_1 = sr_1$$

$$s_2c_2 = s_1r_2$$

$$\dots$$

$$s_mc_m = s_{m-1}r_m$$

$$s = s_mrs.$$

A monoid S is called *left almost regular* if all its elements are left almost regular.

Also recall from [3] that a right S-act A_S satisfies *Condition* (PWP_e) if $ae = a'e$, for $a, a' \in A_S$ and $e \in E(S)$, implies that there exist $a'' \in A_S$ and $u, v \in S$, such that $a = a''u$, $a' = a''v$ and $ue = ve$. It is obvious that Condition (PWP) implies Condition (PWP_e). Also, for idempotent monoids, Conditions (PWP) and (PWP_e) coincide and if $E(S) = \{1\}$, then all right S-acts satisfy Condition (PWP_e). If $S = (\mathbb{N}, .)$ be the monoid of natural numbers with multiplication, then, by Proposition 2.7, there exists at least a right S-act A_S which does not satisfy Condition $(G\text{-}PWP)$. But A_S satisfies Condition (PWP_e), because $E(S) = \{1\}$. So in general Condition (PWP_e) does not imply Condition $(G\text{-}PWP)$.

The following proposition shows that for a (right) left almost regular monoid S Conditions (PWP), $(G\text{-}PWP)$ of (left) right S-acts are equivalent to torsion freeness and Condition (PWP_e) of them. That is,

$$(PWP) \Longleftrightarrow (G\text{-}PWP) \Longleftrightarrow TF \wedge (PWP_e)$$

Proposition 2.24. *Let S be a left almost regular monoid. Then for a right S-act A_S, the following statements are equivalent:*

(1) *A_S satisfies Condition (PWP);*

(2) *A_S satisfies Condition $(G\text{-}PWP)$;*

(3) *A_S is torsion free and satisfies Condition (PWP_e).*

Proof. Implication (1) ⇒ (2) is obvious.
(2) ⇒ (3): Suppose that A_S satisfies Condition $(G\text{-}PWP)$. Then, obviously, A_S is torsion free. Now let $ae = a'e$, for $a, a' \in A_S$ and $e \in E(S)$. Then there exist $a'' \in A_S$, $u, v \in S$ and $n \in \mathbb{N}$ such that $a = a''u$, $a' = a''v$ and $ue^n = ve^n$. The last equality implies that $ue = ve$, and so A_S satisfies Condition (PWP_e).
(3) ⇒ (1): Let A_S be a torsion free right S-act which satisfies Condition (PWP_e). Let $as = a's$, for $a, a' \in A_S$ and $s \in S$. Since S is left almost regular, there exist elements $r, r_1, ..., r_m, s_1, ..., s_m \in S$ and right cancellable elements $c_1, ..., c_m \in S$ such that

$$s_1c_1 = sr_1$$

$$s_2c_2 = s_1r_2$$

$$\cdots$$

$$s_mc_m = s_{m-1}r_m$$

$$s = s_m rs.$$

Hence

$$as_1c_1 = asr_1 = a'sr_1 = a's_1c_1,$$

and so $as_1 = a's_1$. Also,

$$as_2c_2 = as_1r_2 = a's_1r_2 = a's_2c_2,$$

which implies that $as_2 = a's_2$. Continuing this procedure, we obtain that $as_i = a's_i$, for $1 \leq i \leq m$. On the other hand we have

$$s_1c_1 = sr_1 = s_m rsr_1 = s_m rs_1c_1 \;\Rightarrow\; s_1 = s_m rs_1.$$

Continuing this procedure, we have $s_m = s_m rs_m$ and so $e = s_m r$ is an idempotent. Now the equality $as_m = a's_m$ implies that $as_m r = a's_m r$, that is, $ae = a'e$ and so there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $ue = ve$. The last equality implies that $ues = ves$, that is, $us = vs$ and so A_S satisfies Condition (PWP), as required. □

3 Characterization by condition $(G\text{-}PWP)$ on diagonal acts

Here we give a characterization of monoids coming from some special classes, by Condition $(G\text{-}PWP)$ of their diagonal acts. The right S-act $S \times S$ equipped with the right S-action $(s,t)u = (su, tu)$, $s, t, u \in S$ is called the *diagonal act* of monoid S and is denoted by $D(S)$.

Let S be a monoid and $s \in S$. Define

$$L(s, s) = \{(u, v) \in D(S) | us = vs\}.$$

It is obvious that $L(s, s)$ is a left S-act.

Proposition 3.1. *For any monoid S, the following statements are equivalent:*

(1) *for any non-empty set I, $(S^I)_S$ satisfies Condition $(G\text{-}PWP)$;*

(2) $(\forall s \in S)(\exists u, v \in S, n \in \mathbb{N})\ L(s, s) \subseteq S(u, v) \subseteq L(s^n, s^n)$.

Proof. (1) $\Rightarrow$ (2): Suppose that S^I satisfies Condition (*G-PWP*) for any non-empty set I and let $s \in S$. It is obvious that $(s, s) \in L(s, s)$ and so $L(s, s) \neq \emptyset$. Thus we can assume that $L(s, s) = \{(x_i, y_i) | i \in I\}$, where $x_i s = y_i s$, for $i \in I$, thus $(x_i)_I s = (y_i)_I s$ in $(S^I)_S$ and so, by assumption, there exist $(w_i)_I \in (S^I)_S$, $u, v \in S$ and $n \in \mathbb{N}$ such that $(x_i)_I = (w_i)_I u$, $(y_i)_I = (w_i)_I v$ and $us^n = vs^n$. Hence $(x_i, y_i) = w_i(u, v)$, for $i \in I$, which implies that $(x_i, y_i) \in S(u, v)$, for $i \in I$. Thus $L(s, s) \subseteq S(u, v)$. On the other hand the equality $us^n = vs^n$ implies that $(u, v) \in L(s^n, s^n)$, and so $S(u, v) \subseteq L(s^n, s^n)$.
(2) $\Rightarrow$ (1): Let $(x_i)_I s = (y_i)_I s$, for $(x_i)_I, (y_i)_I \in (S^I)_S$ and $s \in S$. Then there exist $u, v \in S$ and $n \in \mathbb{N}$ such that

$$L(s, s) \subseteq S(u, v) \subseteq L(s^n, s^n).$$

The equality $x_i s = y_i s$, $i \in I$, implies that $(x_i, y_i) \in L(s, s)$, $i \in I$ and so there exist $w_i \in S$, $i \in I$, such that $(x_i, y_i) = w_i(u, v)$. That is, $x_i = w_i u$ and $y_i = w_i v$, $i \in I$. Thus $(x_i)_I = (w_i)_I u$ and $(y_i)_I = (w_i)_I v$. Since $(u, v) \in S(u, v) \subseteq L(s^n, s^n)$, we have $us^n = vs^n$ and so $(S^I)_S$ satisfies Condition (*G-PWP*), as required. □

Corollary 3.2. *For any monoid S, the following statements are equivalent:*

(1) *for any non-empty set I, $(S^I)_S$ satisfies Condition (PWP);*

(2) *for every $s \in S$, $L(s, s)$ is a cyclic left S-act.*

Proof. Apply Proposition 3.1, for $n = 1$. □

Proposition 3.3. *For any monoid S, the following statements are equivalent:*

(1) *for every $k \in \mathbb{N}$, $(S^k)_S$ satisfies Condition (G-PWP);*

(2) *$D(S)$ satisfies Condition (G-PWP);*

(3) $(\forall s \in S)(\forall k \in \mathbb{N})(\forall (x_i, y_i) \in L(s, s),\ 1 \leq i \leq k)(\exists u, v \in S)(\exists n \in \mathbb{N})$

$$((x_i, y_i) \in S(u, v) \subseteq L(s^n, s^n),\ 1 \leq i \leq k);$$

(4) $(\forall s \in S)(\forall (x_1, y_1), (x_2, y_2) \in L(s,s))(\exists u, v \in S)(\exists n \in \mathbb{N})$

$$((x_i, y_i) \in S(u,v) \subseteq L(s^n, s^n),\ 1 \leq i \leq 2).$$

Proof. Implications (1) ⇒ (2) and (3) ⇒ (4) are obvious.
(2) ⇒ (4): Suppose that $D(S)$ satisfies Condition (G-PWP) and let

$$(x_1, y_1), (x_2, y_2) \in L(s,s),$$

for $x_1, y_1, x_2, y_2, s \in S$. Then $x_1 s = y_1 s$ and $x_2 s = y_2 s$, which imply that $(x_1, x_2)s = (y_1, y_2)s$. Thus, by assumption, there exist $w_1, w_2, u, v \in S$ and $n \in \mathbb{N}$ such that

$$(x_1, x_2) = (w_1, w_2)u,\ (y_1, y_2) = (w_1, w_2)v,\ us^n = vs^n$$

$$\Longrightarrow x_1 = w_1 u,\ y_1 = w_1 v,\ x_2 = w_2 u,\ y_2 = w_2 v.$$

Thus we have

$$(x_i, y_i) = w_i(u,v) \in S(u,v) \subseteq L(s^n, s^n),\ i = 1, 2.$$

(3) ⇒ (1): Let $(x_1, x_2, ..., x_k)s = (y_1, y_2, ..., y_k)s$, where $x_i, y_i \in S$, $1 \leq i \leq k$. Then $(x_i, y_i) \in L(s,s)$, $1 \leq i \leq k$, and so, by assumption, there exist $u, v \in S$ and $n \in \mathbb{N}$ such that

$$(x_i, y_i) \in S(u,v) \subseteq L(s^n, s^n),\ 1 \leq i \leq k.$$

Thus there exists $w_i \in S$ such that

$$(x_i, y_i) = w_i(u,v),\ us^n = vs^n,\ 1 \leq i \leq k,$$

and so

$$(x_1, x_2, ..., x_k) = (w_1, w_2, ..., w_k)u, (y_1, y_2, ..., y_k) = (w_1, w_2, ..., w_k)v, us^n = vs^n.$$

Hence $(S^k)_S$ satisfies Condition (G-PWP), as required.
(4) ⇒ (3): Let $s \in S$ and $k \in \mathbb{N}$.
If $k = 1$ and $(x_1, y_1) \in L(s,s)$, then $x_1 s = y_1 s$. Since $x_1 = 1x_1$ and $y_1 = 1y_1$, we have

$$(x_1, y_1) \in S(x_1, y_1) \subseteq L(s,s).$$

If $k = 2$, then it is true, by assumption.
Now let $k > 2$, and suppose the assertion is valid for every value less than k. Suppose also that $(x_i, y_i) \in L(s, s)$, for $1 \le i \le k$. Then $(x_i, y_i) \in L(s, s)$, for $1 \le i < k$ imply that there exist $w_1, w_2 \in S$ and $n_1 \in \mathbb{N}$, such that $(x_i, y_i) \in S(w_1, w_2) \subseteq L(s^{n_1}, s^{n_1})$, $1 \le i < k$. On the other hand, since $(x_{k-1}, y_{k-1}), (x_k, y_k) \in L(s, s)$, there exist $w_1^*, w_2^* \in S$ and $n_1^* \in \mathbb{N}$ such that

$$(x_{k-1}, y_{k-1}), (x_k, y_k) \in S(w_1^*, w_2^*) \subseteq L(s^{n_1^*}, s^{n_1^*}).$$

First we suppose that $n_1^* \le n_1$. Then obviously, $L(s^{n_1^*}, s^{n_1^*}) \subseteq L(s^{n_1}, s^{n_1})$, which implies that

$$(w_1, w_2), (w_1^*, w_2^*) \in L(s^{n_1}, s^{n_1}).$$

By assumption, there exist $u, v \in S$ and $n \in \mathbb{N}$ (obviously $n_1 \le n$) such that

$$(w_1, w_2), (w_1^*, w_2^*) \in S(u, v) \subseteq L(s^n, s^n).$$

Thus $S(w_1, w_2) \cup S(w_1^*, w_2^*) \subseteq S(u, v) \subseteq L(s^n, s^n)$, and so

$$(x_i, y_i) \in S(u, v) \subseteq L(s^n, s^n), \ 1 \le i \le k.$$

A similar argument can be used if $n_1 \le n_1^*$. □

Recall that a right S-act A_S is *locally cyclic* if every finitely generated subact of A_S is contained within a cyclic subact of A_S.

Corollary 3.4. *For any monoid S, the following statements are equivalent:*

(1) *for every $k \in \mathbb{N}$, $(S^k)_S$ satisfies Condition (PWP);*

(2) *$D(S)$ satisfies Condition (PWP);*

(3) *for every $s \in S$, $L(s, s)$ is locally cyclic.*

Proof. Apply Proposition 3.3, for $n = 1$. □

Proposition 3.5. *Let S be a commutative monoid. Then, the following statements are equivalent:*

(1) *$D(S)$ satisfies Condition (PWP);*

(2) *$D(S)$ satisfies Condition (G-PWP);*

(3) *S is cancellative.*

Proof. Implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are obvious.
(2) $\Rightarrow$ (3): Let $xc = yc$, for $x, y, c \in S$. Then $(1, x)c = (1, y)c$ in $D(S)$, and so there exist $a, b, u, v \in S$ and $n \in \mathbb{N}$, such that $(1, x) = (a, b)u$, $(1, y) = (a, b)v$ and $uc^n = vc^n$. Thus $x = bu$, $y = bv$ and $au = av = 1$ and so

$$x = bu = b1u = bavu = bvau = y1 = y.$$

Thus S is a right cancellative monoid, as required. □

Proposition 3.6. *For any monoid S, the following statements are equivalent:*

(1) *$D(S)$ satisfies Condition (PWP) and $|E(S)| \leq 2$;*

(2) *$D(S)$ satisfies Condition (G-PWP) and $|E(S)| \leq 2$;*

(3) *S is right cancellative.*

Proof. Implication (1) $\Rightarrow$ (2) is obvious.
(2) $\Rightarrow$ (3): Let $xc = yc$, for $x, y, c \in S$. Then $(1, x)c = (1, y)c$ in $D(S)$. Since $D(S)$ satisfies Condition (*G-PWP*), there exist $a, b, u, v \in S$ and $n \in \mathbb{N}$, such that $(1, x) = (a, b)u$, $(1, y) = (a, b)v$ and $uc^n = vc^n$. Thus $au = av = 1$, and so ua and va are idempotents. If $ua = va$, then $uau = vau$ and so $u = v$. Thus $x = bu = bv = y$. If $ua \neq va$, then either $ua = 1$ or $va = 1$. For example if $ua = 1$, then we have $v = 1v = uav = u1 = u$, and so $x = bu = bv = y$. Thus S is a right cancellative monoid, as required.
(3) $\Rightarrow$ (1): If S is right cancellative, then obviously $D(S)$ satisfies Condition (*PWP*) and so $|E(S)| = 1$. □

Proposition 3.7. *For an idempotent monoid S, the following statements are equivalent:*

(1) *$D(S)$ satisfies Condition (PWP);*

(2) *$D(S)$ satisfies Condition (G-PWP);*

(3) *$S = \{1\}$.*

Proof. Implications (1) ⇒ (2) and (3) ⇒ (1) are obvious.
(2) ⇒ (3): Let $s \in S$. Then $(1, s)s = (s, 1)s$ in $D(S)$. Since $D(S)$ satisfies Condition $(G\text{-}PWP)$ there exist $a, b, u, v \in S$ and $n \in \mathbb{N}$ such that $(1, s) = (a, b)u$, $(s, 1) = (a, b)v$ and $us^n = vs^n$. Thus $1 = au$ and so $a = u = 1$. Similarly, $v = 1$, and so $s = av = 1$. That is, $S = \{1\}$, as required. □

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