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On the pointfree counterpart of the local definition of classical continuous maps

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Abstract. The familiar classical result that a continuous map from a space X to a space Y can be defined by giving continuous maps $\varphi_U : U \to Y$ on each member U of an open cover $\mathfrak{C}$ of X such that $\varphi_U \mid U \cap V = \varphi_V \mid U \cap V$ for all $U, V \in \mathfrak{C}$ was recently shown to have an exact analogue in pointfree topology, and the same was done for the familiar classical counterpart concerning *finite closed* covers of a space X (Picado and Pultr [4]). This note presents alternative proofs of these pointfree results which differ from those of [4] by treating the issue in terms of *frame homomorphisms* while the latter deals with the dual situation concerning *localic maps*. A notable advantage of the present approach is that it also provides proofs of the analogous results for some significant variants of frames which are not covered by the localic arguments.

A continuous map $\varphi : X \to Y$ between topological spaces is said to be *locally defined* if one is given an open cover $\mathfrak{C}$ of X and a continuous map $\varphi_U : U \to Y$ for each $U \in \mathfrak{C}$ such that $\varphi_U \mid U \cap V = \varphi_V \mid U \cap V$ for all $U, V \in \mathfrak{C}$, and φ is then taken as the common extension of the φ_U, $U \in \mathfrak{C}$, whose existence is

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ensured by these conditions and whose continuity readily follows from them. Now, quite remarkably, as shown in [4], there is an exact counterpart of this notion in *pointfree* topology, that is, in the category of frames, despite the (possible) absence of points there which seem to play a crucial rôle here. The purpose of this note is to present an alternative to the proof given loc. cit.; specifically, while the latter uses arguments about sublocales, it will be shown here that surprisingly simple considerations involving only frame homomorphisms also prove the result in question. As a particular aspect of this approach, we note that the arguments used here apply equally well to the κ-*frames* for any regular cardinal κ (and so in particular to the bounded distributive lattices (= ω-frames) and the σ-frames (= ω_1-frames) and to the *preframes* – which is clearly not the case for the earlier proofs.

Concerning background, we only require familiarity with the most basic notions concerning frames and their homomorphism, as presented in the introductory parts of Picado and Pultr [4]. Specifically, a cover of a frame L is a subset C of L whose joint $\bigvee C$ equals e, the top of L; $\prod$ stands for the usual product of frames; for any element a of a frame L, ${\downarrow}a = \{s \in L \mid s \leq a\}$ which is the image of L by the frame homomorphism $(\cdot) \wedge a : L \to {\downarrow}a$, $s \mapsto s \wedge a$, and similarly for ${\uparrow}a = \{s \in L \mid s \geq a\}$ and $(\cdot) \vee a : L \to {\uparrow}a$, $s \mapsto s \vee a$. Regarding the relation between spaces and frames, $\mathfrak{O}$ is the usual functor so that $\mathfrak{O}X$ is the frame of all open subsets U of X, and for any continuous map $\varphi : X \to Y$ between spaces $\mathfrak{O}\varphi : \mathfrak{O}Y \to \mathfrak{O}X$ is the frame homomorphism $U \mapsto \varphi^{-1}[U]$.

The first step towards the result is the following which seems to be new although it really is quite obvious.

Lemma 1.1. *For any cover C of a frame L, if*

$$N = \{u = (u_a)_{a\in C} \in \prod\{{\downarrow}a \mid a \in C\} \mid u_a \wedge a \wedge b = u_b \wedge a \wedge b \text{ for all } a, b \in C\}$$

then $k : L \to N$, $s \mapsto (s \wedge a)_{a\in C}$, is an isomorphism.

Proof. Each $k(s)$ obviously belongs to N which, in turn, is a subframe of $\prod\{{\downarrow}a \mid a \in C\}$ so that k is indeed a homomorphism, by the familiar properties of products of frames, trivially one-one since C is a cover of L. Further, k is onto: for any $u \in N$ and $a, b \in C$, $u_a \wedge b \leq a \wedge b$ so that

$$u_a \wedge b = u_a \wedge b \wedge a \wedge b = u_b \wedge a \wedge b \leq u_b,$$

the second step by the definition of N, and hence

$$k\left(\bigvee\{u_a \mid a \in C\}\right) = \left(\left(\bigvee\{u_a \mid a \in C\}\right) \wedge b\right)_{b\in C} = (u_b)_{b\in C} = u.$$

□

Proposition 1.2. *For any frames L and M, if C is a cover of L and $h_a : M \to\downarrow a$ is a homomorphism for each $a \in C$ such that*

$$h_a(s) \wedge a \wedge b = h_b(s) \wedge a \wedge b \qquad (a, b \in C,\ s \in M)$$

then there exists a unique frame homomorphism $h : M \to L$ such that $h(s) \wedge a = h_a(s)$ for all $a \in C$ and $s \in M$.

Proof. Note that the homomorphism

$$M \to \prod\{\downarrow a \mid a \in C\},\ s \mapsto (h_a(s))_{a\in C},$$

actually maps into the N of Lemma 1.1 by the given condition on the h_a, and the corresponding homomorphism $f : M \to N$ then determines $h = k^{-1}f$, with k^{-1} provided by Lemma 1.1. Consequently, $f = kh$ so that

$$(h_a(s))_{a\in C} = f(s) = kh(s) = (h(s) \wedge a)_{a\in C},$$

as claimed, and this condition clearly makes h unique. □

To put the above in perspective it may be useful to see what the present approach amounts to in the classical situation. Consider, then, topological spaces X and Y with an open cover $\mathfrak{C}$ of X and a continuous map $\varphi_U : U \to Y$ for each $U \in \mathfrak{C}$ such that $\varphi_U \mid U \cap V = \varphi_V \mid U \cap V$ for all $U, V \in \mathfrak{C}$. Then, the usual functor $\mathfrak{O}$ from spaces to frames determines a cover $\mathfrak{C}$ of the frame $\mathfrak{O}X$ and frame homomorphisms

$$\mathfrak{O}\varphi_U : \mathfrak{O}Y \to \mathfrak{O}U = \downarrow U\ (\subseteq \mathfrak{O}X)$$

for each $U \in \mathfrak{C}$ such that

$$\begin{aligned}(\mathfrak{O}\varphi_U)(W) \cap U \cap V &= \varphi_U^{-1}[W] \cap U \cap V\\ &= \{x \in U \cap V \mid \varphi_U(x) \in W\}\\ &= \{x \in U \cap V \mid \varphi_V(x) \in W\}\\ &= (\mathfrak{O}\varphi_V)(W) \cap U \cap V,\end{aligned}$$

the second step exactly because $\varphi_U \mid U \cap V = \varphi_V \mid U \cap V$, and the proposition now supplies a unique frame homomorphism $h : \mathfrak{O}Y \to \mathfrak{O}X$ such that $U \cap h[W] = \varphi_U^{-1}[W]$ for all $U \in \mathfrak{C}$ and $W \in \mathfrak{O}Y$. On the other hand, for the common extension φ of the φ_U, just taken as a *set map* from X to Y,

$$\begin{aligned} U \cap h(W) = \varphi_U^{-1}[W] &= \{x \in U \mid \varphi_U(x) \in W\} \\ &= \{x \in U \mid \varphi(x) \in W\} = U \cap \varphi^{-1}[W], \end{aligned}$$

for all $U \in \mathfrak{C}$ and $W \in \mathfrak{O}Y$ so that $\varphi^{-1}[W] = h[W]$, showing φ is continuous and $h = \mathfrak{O}\varphi$. In particular, the classical result in question is thus seen to be a consequence of its pointfree counterpart.

Remark 1.3. Considering the above in categorical terms, it is clear by its very definition (see, for instance, Mac Lane [2]) that the frame N, together with the homomorphisms $N \to {\downarrow}a$ induced by the product projections of $\prod\{{\downarrow}a \mid a \in C\}$, provides the limit of the diagram

$$\begin{array}{ccc} {\downarrow}a & & \\ & \searrow & \\ & & {\downarrow}(a \wedge b) \qquad (a, b \in C), \\ & \nearrow & \\ {\downarrow}b & & \end{array}$$

and by Lemma 1.1 the same then holds for L and the maps $(\cdot) \wedge a : L \to {\downarrow}a$, $a \in C$. Of course, this hardly adds anything to the result as such but it still seems worth pointing out.

Remark 1.4. Concerning Proposition 1.2, the desired $h : M \to L$ is obviously given by $h(s) = \bigvee\{h_a(s) \mid a \in C\}$, and without Lemma 1.1 one could just consider this map $M \to L$ and show (i) it is a homomorphism such that (ii) $a \wedge h(s) = h_a(s)$ for all $a \in C$ and $s \in M$. In the end, though, this seems to require no less work than the route chosen above while the latter provides a much clearer picture of what is really going on.

Remark 1.5. Regarding the earlier comment that the arguments used here are all equally applicable to the κ-frames for any regular cardinal κ, it should be pointed out that, in this situation, the covers involved are of course those provided by the given setting, that is, the *κ-small* subsets C such that $\bigvee C = e$: this then trivially implies that all the other joins which enter into the argument exist. Other than that, it is sufficient to observe that all the entities involved in the above proof are κ-frames and κ-frame homomorphisms whenever this is the case for the initial data to see that the counterparts of Lemma 1.1 and Proposition 1.2 hold.

Further, the analogous situation arises in the case of *preframes*, albeit somewhat less obviously. To see this, recall that a preframe is a partially ordered set in which all finitary meets and updirected joins exist such that the binary meet distributes over the latter [1], and a cover is then understood to be an *updirect* subset C such that $\bigvee C = e$. Now, for any $(u_a)_{a\in C}$ in $\prod\{{\downarrow}a \mid a \in C\}$ such that $u_a \wedge a \wedge b = u_b \wedge a \wedge b$, $u_a \leq u_b$ whenever $a \leq b$ since $u_a \leq u_a \wedge a \wedge b = u_b \wedge a \wedge b \leq u_b$, and hence $\{u_a \mid a \in C\}$ is updirected since C is. Thus, the joins in the relevant proof are indeed updirected so that they do exist. Moreover, the entities used in the above proofs are again readily seen to belong to the category of preframes provided this holds for the initial data, and the desired results then follow.

We now turn to the related classical situation which involves *finite closed covers* of topological spaces also considered in [4]: if $X = S_1 \cup \cdots \cup S_n$ for closed $S_i \subseteq X$ and $\varphi_i : S_i \to Y$ is a continuous map for each $i = 1, \ldots, n$ such that $\varphi_i \mid S_i \cap S_k = \varphi_k \mid S_i \cap S_k$ for all $i, k = 1, \ldots, n$, the common extension of $\varphi : X \to Y$ of these φ_i is continuous. As shown loc. cit., this also has a pointfree counterpart, and here we provide a new proof in the same spirit as that given above.

Again, there is an obvious basic lemma which can then be applied to obtain the proposition in question. As is to be expected, the present setting deals with finite subsets F of a frame L such that $\bigwedge F = 0$, called the *finite cocovers* of L, and the results are now as follows.

Lemma 1.6. *For any finite cocover F of a frame L, if*

$$N = \{u = (u_a)_{a\in F} \in \prod\{{\uparrow}a \mid a \in F\} \mid u_a \vee a \vee b = u_b \vee a \vee b \textit{ for all } a, b \in F\}$$

then $k : L \to N$, $s \mapsto (s \vee a)_{a\in F}$, is an isomorphism.

Proposition 1.7. *For any frames L and M, if F is a finite cocover of L and $h_a : M \to {\uparrow}a$ is a homomorphism for each $a \in F$ such that*

$$h_a(s) \vee a \vee b = h_b(s) \vee a \vee b \qquad (a, b \in F,\ s \in M)$$

then there exists a unique frame homomorphism $h : M \to L$ *such that* $h(s) \vee a = h_a(s)$ *for all* $s \in M$ *and* $a \in F$*.*

Proof. The proofs here are exact copies of the earlier ones, with the modification that joins and meets are interchanged. We omit the details.

To see that the frame situation considered here indeed corresponds to the spatial case involving finite closed covers, let $X = S_1 \cup \cdots \cup S_n$ and $\varphi_i : S_i \to Y$ be as described earlier. Then this determines frame homomorphisms

$$h_i : \mathfrak{O}Y \to {\uparrow}\mathbb{C}S_i, \ W \mapsto \varphi_i^{-1}[W] \cup \mathbb{C}S_i,$$

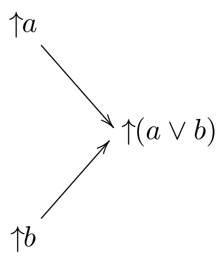
where $\mathbb{C}S_i = X \backslash S_i$ and ${\uparrow}\mathbb{C}S_i$ is taken in $\mathfrak{O}X$ while h_i is based on the familiar fact that $\mathfrak{O}S \cong {\uparrow}\mathbb{C}S$ for any closed subspace S of X. Further $\{\mathbb{C}S_1, \ldots \mathbb{C}S_n\}$ is clearly a finite cocover of $\mathfrak{O}X$.

Now, for any $x \in \varphi_i^{-1}[W] \cup \mathbb{C}S_i \cup \mathbb{C}S_k$, if $x \notin \mathbb{C}S_i \cup \mathbb{C}S_k$ then $x \in S_i \cap S_k$ so that $\varphi_i(x) = \varphi_k(x)$ by hypothesis; hence $\varphi_i^{-1}[W] \cup \mathbb{C}S_i \cup \mathbb{C}S_k \subseteq \varphi_k^{-1}[W] \cup \mathbb{C}S_i \cup \mathbb{C}S_k$ and by symmetry this implies equality, showing that the h_i satisfy the condition given in the above proposition. Thus there exists a unique frame homomorphism $h : \mathfrak{O}Y \to \mathfrak{O}X$ such that

$$h(W) \cup \mathbb{C}S_i = h_i(W) = \varphi_i^{-1}[W] \cup \mathbb{C}S_i \qquad (i = 1, \ldots, n).$$

On the other hand, for the common set map extension φ of the φ_i, a simple calculation shows that $\varphi_i^{-1}[W] \cup \mathbb{C}S_i = \varphi^{-1}[W] \cup \mathbb{C}S_i$ and hence $h(W) \cup \mathbb{C}S_i = \varphi^{-1}[W] \cup \mathbb{C}S_i$ for all i; consequently $\varphi^{-1}[W] = h(W)$, saying φ is continuous and $h = \mathfrak{O}\varphi$, as desired. Thus, again, the classical fact involved here turns out to follow from the present pointfree result. $\square$

Remark 1.8. Here we have the exact counterpart of Remark 1.3, with the diagram in question now given by

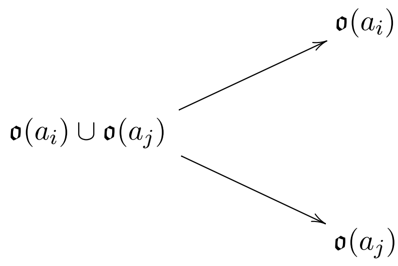


$(a, b \in F).$

Remark 1.9. Clearly, the comment in Remark 1.4 concerning Proposition 1.2 equally well applies here: the use of Lemma 1.6 can readily be avoided to obtain Proposition 1.7 but the route chosen here seemed to provide deeper insight.

Remark 1.10. As in the earlier situation, the proofs of Lemma 1.6 and Proposition 1.7 can quite obviously be adjusted to provide proofs of the κ-frame counterparts of these results. On the other hand, this is clearly *not* the case concerning preframes.

We conclude with some remarks comparing the present treatment of the topic with that of [4]. Clearly Proposition 1.2 is the exact translation of Theorem 2.6 in [4] from the language of sublocales into that of frame quotients so that the two are essentially the same although they do, of course, differ in flavour. Specifically, the situation Proposition 1.2 deals with is the direct image of the classical situation in question if one regards the usual functor $\mathfrak{O}$ as the guide: a continuous map $\varphi : U \to Y$ corresponds to the frame homomorphism $\mathfrak{O}\varphi : \mathfrak{O}Y \to \mathfrak{O}U$ where the latter is the open quotient $\downarrow U$ of $\mathfrak{O}X$ and not the corresponding open sublocale of $\mathfrak{O}X$. As a consequence, deriving the classical result in question from Theorem 2.6 of [4] requires an appropriate translation from open sublocales to open quotients which is not the case for Proposition 1.2, as shown earlier. Other than that, there is of course the fundamental difference between the proofs in [4] and here, where the crucial step is Lemma 1.1 which in essence trivializes Proposition 1.2. There is a hint at the isomorphism of Lemma 1 in [4], given by the comment 2.7 that the inclusion maps $\mathfrak{o}(a_i) \to L$ provide the locale colimit of the diagram



but, of course, this appears as an afterthought rather than as a step of the proof.

Regarding Proposition 1.7, it is clear that the finite cocovers F of a frame L correspond exactly to the covers of L by the closed sublocales $\uparrow a$, $a \in F$, and Proposition 1.7 then says precisely the same as Theorem 3.4 of [4], expressed in terms of frame homomorphisms. Thus, the translation from sublocales to frame quotients is rather more direct here than in the earlier case. Of course, the proofs in question are formally different: that of Proposition 1.7 does not depend on the inductive step implicitly used in the proof of Theorem 3.4 which the simple direct argument avoids, and there is the difference between using Lemma 1.6 and arguing at the level of the relevant pushout maps, similar to the situation in the earlier case.

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