



On finitely generated modules whose first nonzero Fitting ideals are regular

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Abstract. A finitely generated R -module is said to be a module of type (F_r) if its $(r - 1)$ -th Fitting ideal is the zero ideal and its r -th Fitting ideal is a regular ideal. Let R be a commutative ring and N be a submodule of R^n which is generated by columns of a matrix $A = (a_{ij})$ with $a_{ij} \in R$ for all $1 \leq i \leq n, j \in \Lambda$, where Λ is a (possibly infinite) index set. Let $M = R^n/N$ be a module of type (F_{n-1}) and $\mathbf{T}(M)$ be the submodule of M consisting of all elements of M that are annihilated by a regular element of R . For $\lambda \in \Lambda$, put $M_\lambda = R^n / \langle (a_{1\lambda}, \dots, a_{n\lambda})^t \rangle$. The main result of this paper asserts that if M_λ is a regular R -module, for some $\lambda \in \Lambda$, then $M/\mathbf{T}(M) \cong M_\lambda/\mathbf{T}(M_\lambda)$. Also it is shown that if M_λ is a regular torsionfree R -module, for some $\lambda \in \Lambda$, then $M \cong M_\lambda$. As a consequence we characterize all non-torsionfree modules over a regular ring, whose first nonzero Fitting ideals are maximal.

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Keywords: Fitting ideals, type of a module, torsion submodule.

Mathematics Subject Classification [2010]: 13C05, 13C10, 13C12.

Received: 8 May 2016, Accepted: 15 September 2016

ISSN Print: 2345-5853 Online: 2345-5861

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1 Introduction and Preliminaries

Let R be a commutative ring with identity and M be a finitely generated R -module. For a set $\{x_1, \dots, x_n\}$ of generators of M there is an exact sequence

$$0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0, \quad (1)$$

where R^n is a free R -module with the set $\{e_1, \dots, e_n\}$ of basis, the R -homomorphism φ is defined by $\varphi(e_j) = x_j$ and N is the kernel of φ . Let N be generated by $u_\lambda = a_{1\lambda}e_1 + \dots + a_{n\lambda}e_n$, with λ in some index set Λ . Assume that A be the following matrix:

$$\begin{pmatrix} \dots & a_{1\lambda} & \dots \\ \vdots & \vdots & \vdots \\ \dots & a_{n\lambda} & \dots \end{pmatrix}.$$

(We call A the matrix presentation of the sequence (1)). Let $\text{Fitt}_i(M)$ be an ideal of R generated by the minors of size $n - i$ of matrix A . For $i \geq n$, $\text{Fitt}_i(M)$ is defined R and for $i < 0$, $\text{Fitt}_i(M)$ is defined as the zero ideal. It is known that $\text{Fitt}_i(M)$ is the invariant ideal determined by M , that is, it is determined uniquely by M and it does not depend on the choice of the set of generators of M [4]. The ideal $\text{Fitt}_i(M)$ will be called the i -th Fitting ideal of the module M . It follows from the definition that $\text{Fitt}_i(M) \subseteq \text{Fitt}_{i+1}(M)$, for every i . The most important Fitting ideal of M is the first of the $\text{Fitt}_i(M)$ that is nonzero. We shall denote this Fitting ideal by $I(M)$.

An element of R is regular if it is a nonzerodivisor and an ideal of R is regular if it contains a regular element. Assume that $\mathbf{T}(M)$, the torsion submodule of M , be the submodule of M consisting of all elements of M that are annihilated by a regular element of R . An R -module M is a torsion module if $M = \mathbf{T}(M)$ and is a torsionfree R -module if $\mathbf{T}(M) = 0$.

One of the most interesting question for modules is, "Can we characterize modules according to the first nonzero Fitting ideal of them?" D.A. Buchsbaum and D. Eisenbud have shown in [2] that if R is a Noetherian ring then, M is a finitely generated projective R -module of constant rank if and only if $I(M) = R$. Also a lemma of J. Lipman asserts that if R is a quasilocal ring and $M = R^n/K$, where K is a submodule of R^n , then $I(M)$ is regular principal if and only if K is finitely generated free and $M/T(M)$ is free of

rank $n - q$ ([6]) and J. Ohm generalized this result to global case [7]. At this point, a natural question arises: if $I(M)$ is any ideal, how much can we say about the structure of M ? In this paper this question is partly answered.

2 The main results

A finitely generated module is said to be a module of type (F_r) if its $(r-1)$ -th Fitting ideal is the zero ideal and its r -th Fitting ideal is a regular ideal.

Let M be a finitely generated R -module. Then there exist some $n \in \mathbb{N}$ and a submodule N of R^n such that $M \cong R^n/N$. The purpose of this paper is to study the module of type (F_{n-1}) .

Lemma 2.1. *Let R be a Noetherian ring and M be a finitely generated R -module of type (F_r) . Then exactly one of the following holds:*

- (1) M is projective of constant rank r .
- (2) M can not be generated by r elements.

Proof. Let M can be generated by r elements. So, by definition we have $\text{Fitt}_r(M) = R$. Since M is a module of type (F_r) , so $\text{Fitt}_{r-1}(M) = 0$. Hence by [2, Lemma 1], M is projective of constant rank r . \square

Let (R, m) be a local ring and M be a finitely generated R -module. It is known that all minimal generator sets of M have the same cardinal. We will denote the minimal number of generators of M by $\mu(M)$.

Example 2.2. Let (R, m) be a local ring and m be a finitely generated regular ideal of R . Let M be a finitely generated module with $\mu(M) = n$ and $I(M) = m$. Then M is a module of type (F_{n-1}) . Because let $X = \{x_1, \dots, x_n\}$ be a minimal generator set of M and

$$0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0$$

be an exact sequence, where R^n is a free R -module with the set $\{e_1, \dots, e_n\}$ of basis, the R -homomorphism φ is defined by $\varphi(e_j) = x_j$ and N is the kernel of φ . Assume that

$$A = \begin{pmatrix} \dots & a_{1\lambda} & \dots \\ \vdots & \vdots & \vdots \\ \dots & a_{n\lambda} & \dots \end{pmatrix}$$

is the matrix presentation of this sequence, with λ in some index set Λ . Since X is a minimal generator set of M , it is easily seen that $a_{i\lambda} \in m$, for every i , $1 \leq i \leq n$ and every $\lambda \in \Lambda$. Let $m = I(M) = \text{Fitt}_i(M)$. Since $a_{i\lambda} \in m$, hence $m = \text{Fitt}_i(M) \subseteq m^{n-i}$. If $n = i$, then $m = \text{Fitt}_n(M) = R$, which is a contradiction. Hence by Nakayama's Lemma $i = n - 1$. So M is a module of type (F_{n-1}) .

An R -module M is called a regular module if $I(M)$ is a regular ideal.

Theorem 2.3. *Let R be a commutative ring and N be a submodule of R^n consisting of elements $A_\lambda = (a_{1\lambda}, \dots, a_{n\lambda})^t$ with λ in some index set Λ . Let $M \cong R^n/N$ be an R -module of type (F_{n-1}) . For $\lambda \in \Lambda$, put $M_\lambda = R^n / \langle (a_{1\lambda}, \dots, a_{n\lambda})^t \rangle$. If M_λ is a regular module, for some $\lambda \in \Lambda$, then $M/\mathbf{T}(M) \cong M_\lambda/\mathbf{T}(M_\lambda)$.*

Proof. Clearly

$$0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0$$

is an exact sequence and

$$A = \begin{pmatrix} \dots & a_{1\lambda} & \dots \\ \vdots & \vdots & \vdots \\ \dots & a_{n\lambda} & \dots \end{pmatrix}$$

is the matrix presentation of this sequence. Let M_l be a regular R -module, for some $l \in \Lambda$. This means that $I(M_l)$ is a regular ideal.

Put $M_l = R^n / \langle A_l \rangle$. Set $x_i = e_i + N$ and $y_i = e_i + \langle A_l \rangle$ for all $1 \leq i \leq n$. Then $M = \langle x_1, \dots, x_n \rangle$ and $M_l = \langle y_1, \dots, y_n \rangle$. Define

$$f : M/\mathbf{T}(M) \longrightarrow M_l/\mathbf{T}(M_l); f\left(\sum_{i=1}^n a_i x_i + \mathbf{T}(M)\right) = \sum_{i=1}^n a_i y_i + \mathbf{T}(M_l).$$

Let $x = \sum_{i=1}^n a_i x_i \in \mathbf{T}(M)$. Hence there exists a regular element q in R such that $qx = \sum_{i=1}^n qa_i x_i = 0$. So there exist some elements $r_k \in R$, $1 \leq k \leq m$ such that $qa_i = \sum_{k=1}^m r_k a_{i\lambda_k}$, $1 \leq i \leq n$, $\lambda_k \in \Lambda$. For every i, j , $1 \leq i, j \leq n$ and every $\lambda \in \Lambda$, we have $qa_i a_{j\lambda} = \sum_{k=1}^m r_k a_{i\lambda_k} a_{j\lambda}$. Thus $q(a_i a_{j\lambda} - a_j a_{i\lambda}) = \sum_{k=1}^m r_k (a_{i\lambda_k} a_{j\lambda} - a_{j\lambda_k} a_{i\lambda})$. Since M is a module of type

(F_{n-1}) , so $\text{Fitt}_{n-2}(M) = 0$. Hence $a_{i\lambda_k}a_{j\lambda} - a_{j\lambda_k}a_{i\lambda} = 0$ and so we have $q(a_ia_{j\lambda} - a_ja_{i\lambda}) = 0$. Since q is regular, hence

$$a_ia_{j\lambda} - a_ja_{i\lambda} = 0. \quad (1)$$

Now put $y = \sum_{i=1}^n a_iy_i$. Since $M_l = R^n / \langle A_l \rangle = \langle y_1, \dots, y_n \rangle$, we have $a_{1l}y_1 + \dots + a_{nl}y_n = 0$. Let $s, 1 \leq s \leq n$ be arbitrary. We have $a_{sl}y_s = -a_{1l}y_1 - \dots - a_{nl}y_n$. Therefore $a_{sl}y = \sum_{i=1}^n a_{sl}a_iy_i = a_{sl}a_sy_s + \sum_{s \neq i=1}^n a_{sl}a_iy_i = \sum_{s \neq i=1}^n -a_s a_{il}y_i + \sum_{s \neq i=1}^n a_{sl}a_iy_i = \sum_{s \neq i=1}^n (a_{sl}a_i - a_{il}a_s)y_i$. Therefore by (2), we have $a_{sl}y = 0$. Since a_{sl} is arbitrary, so $\mathbf{I}(M_l)y = 0$. On the other hand, by hypothesis, $\mathbf{I}(M_l)$ is a regular ideal, hence $y \in \mathbf{T}(M_l)$. Therefore f is welldefined. It is clear that f is onto. Now let $y = \sum_{i=1}^n a_iy_i \in \mathbf{T}(M_l)$. So there exists a regular element $p \in R$ such that $p(a_1, \dots, a_n)^t \in \langle A_l \rangle \subseteq N$. Thus $\sum_{i=1}^n a_ix_i \in \mathbf{T}(M)$. Therefore f is an isomorphism. \square

Proposition 2.4. *Let $M \cong R^n/N$ be an R -module of type (F_{n-1}) . If M_λ is a regular torsionfree module, for some $\lambda \in \Lambda$, then $M = M_\lambda = R^n / \langle (a_{1\lambda}, \dots, a_{n\lambda})^t \rangle$.*

Proof. Let M_λ be a regular torsionfree R -module, for some $\lambda \in \Lambda$. Since M_λ is a regular module, hence by the proof of Theorem 2.3, $x = \sum_{i=1}^n a_ix_i \in \mathbf{T}(M)$ if and only if $\sum_{i=1}^n a_iy_i \in \mathbf{T}(M_\lambda) = 0$. For every $l \in \Lambda$, we have $\sum_{i=1}^n a_ix_i = 0 \in \mathbf{T}(M)$. Hence $\sum_{i=1}^n a_iy_i \in \mathbf{T}(M_\lambda) = 0$. Thus $(a_{1l}, \dots, a_{nl})^t \in \langle A_\lambda \rangle$, for every $l \in \Lambda$. So $N \subseteq \langle A_\lambda \rangle$. This means that $M = M_\lambda$. \square

Corollary 2.5. *Let $M \cong R^n/N$ be a finitely generated R -module of type (F_{n-1}) . If M_λ is a regular torsionfree module, for some $\lambda \in \Lambda$, then $\text{pd}_R(M) = 1$.*

Proof. By Proposition 2.4, we have $M \cong R^n / \langle (a_{1\lambda}, \dots, a_{n\lambda})^t \rangle = M_\lambda$. Since M_λ is a regular module, hence $0 \longrightarrow R \longrightarrow R^n \longrightarrow M \longrightarrow 0$, is a free resolution of M . So $\text{pd}_R(M) = 1$. \square

Lemma 2.6. *Let $M \cong R^n/N$ be a finitely generated non-torsionfree R -module of type (F_{n-1}) . Then $M/\mathbf{T}(M)$ is free of rank $n - 1$ if there exists $\lambda \in \Lambda$ such that $\langle a_{1\lambda}, \dots, a_{n\lambda} \rangle$ is a principal regular ideal.*

Proof. By [7, Theorem 6.2], $\langle a_{1\lambda}, \dots, a_{n\lambda} \rangle$ is a principal regular ideal if and only if $M_\lambda/\mathbf{T}(M_\lambda)$ is free of rank $n - 1$. (Note that if $\langle a_{1\lambda}, \dots, a_{n\lambda} \rangle$ is a regular ideal, then $\langle (a_{1\lambda}, \dots, a_{n\lambda})^t \rangle$ is a free R -module). Therefore by Theorem 2.3, $M/\mathbf{T}(M)$ is free of rank $n - 1$ if there exists some $\lambda \in \Lambda$ such that $\langle a_{1\lambda}, \dots, a_{n\lambda} \rangle$ is a principal regular ideal. \square

Theorem 2.7. *Let (R, m) be a Noetherian local ring and $M \cong R^n/N$ be an R -module of type (F_{n-1}) . Assume that there exists an element $(a_{1l}, \dots, a_{nl})^t \in N$ such that a_{sl} is a regular prime element of R , for some s , $1 \leq s \leq n$. Then exactly one of the following holds:*

- (1) $M \cong R/\mathbf{I}(M) \oplus R^{n-1}$,
- (2) $M \cong R^n / \langle (a_{1l}, \dots, a_{nl})^t \rangle$.

Proof. Clearly

$$0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0$$

is an exact sequence. Assume that

$$A = \begin{pmatrix} \dots & a_{1\lambda} & \dots \\ \vdots & \vdots & \vdots \\ \dots & a_{n\lambda} & \dots \end{pmatrix}$$

be the matrix presentation of this sequence. Put $p = a_{sl}$ and $M_l = R^n / \langle (a_{1l}, \dots, a_{nl})^t \rangle$. We consider two cases:

Case 1) Let $p = a_{sl} \mid a_{il}$, for every i , $1 \leq i \leq n$. Thus $\mathbf{I}(M_l) = \langle a_{1l}, \dots, a_{nl} \rangle = (p)$ is a regular principal ideal. So by Lemma 2.6, $M/\mathbf{T}(M)$ is free of rank $n - 1$. Hence $M \cong \mathbf{T}(M) \oplus R^{n-1}$. If $\mathbf{T}(M) = 0$, then $M \cong R^{n-1}$. Let $\mathbf{T}(M) \neq 0$. Since M is a module of type (F_{n-1}) , so by Lemma 2.1, $\mu(M) = n$. Because R is a local ring, it is easily seen that $n = \mu(M) = \mu(\mathbf{T}(M)) + \mu(R^{n-1}) = \mu(\mathbf{T}(M)) + n - 1$. Hence $\mu(\mathbf{T}(M)) = 1$ and so $\mathbf{T}(M)$ is a cyclic R -module. On the other hand by [1, page 174], $\mathbf{I}(M) = \text{Fitt}_0(\mathbf{T}(M))$. Since $\mathbf{T}(M)$ is a cyclic module, hence $\mathbf{T}(M) \cong R/\text{Fitt}_0(\mathbf{T}(M))$ and so $M \cong R/\mathbf{I}(M) \oplus R^{n-1}$.

Case 2) Now let there exist some t , $1 \leq t \leq n$ such that $p = a_{sl} \nmid a_{tl}$. We claim that $M \cong M_l$, in this case. First we show that $\mathbf{T}(M_l) = 0$. Assume that there exists some regular element $q \in R$ and an element $s' \in R$ such that $q(a_{1l}, \dots, a_{nl})^t = s'(a_{1l}, \dots, a_{nl})^t$, for some $(a_{1l}, \dots, a_{nl})^t \in R^n$. Therefore for $1 \leq i \leq n$, we have

$$qa_i = s'a_{il}. \quad (2)$$

So $qa_s = s'a_{sl} = s'p$. Thus $p \mid qa_s$. Since p is a prime element, then $p \mid q$ or $p \mid a_s$.

Let $p \mid q$. Thus there exists an element $t' \in R$ such that $q = pt'$. So $pt'a_s = s'p$. Since p is regular, one gets that

$$s' = t'a_s. \tag{3}$$

By (3) and (4) we have $pt'a_i = t'a_s a_{il}$, for every i , $1 \leq i \leq n$. Since t' is a regular element, so, for every i ,

$$pa_i = a_s a_{il}. \tag{4}$$

Since $p \nmid a_{tl}$, hence $p \mid a_s$. Therefore there exists $t'' \in R$ such that $a_s = pt''$. By (5), we have $pa_i = pt''a_{il}$. So $a_i = t''a_{il}$, for every i , $1 \leq i \leq n$. Hence $(a_1, \dots, a_n)^t = t''(a_{1l}, \dots, a_{nl})^t$. Thus $\mathbf{T}(M_l) = 0$.

Now, let $p \nmid q$. Then $p \mid a_s$. Thus there exists some $t'' \in R$ such that $a_s = t''p$. From (3), we have $qa_s = s'p$. Thus $qt''p = s'p$. So $s' = qt''$. Again by (3), $qa_i = qt''a_{il}$. Hence $(a_1, \dots, a_n)^t = t''(a_{1l}, \dots, a_{nl})^t$ and so $\mathbf{T}(M_l) = 0$. By Proposition 2.4, we have $M = M_l = R^n / \langle (a_{1l}, \dots, a_{nl})^t \rangle$. \square

3 Modules over regular rings

The Krull dimension of R is the supremum of all lengths of chains of prime ideals of R . Let R be a Noetherian local ring with maximal ideal m and Krull dimension d . Recall that R is called a regular local ring if m has a generating set with d elements. The generating set of d elements for m is called a regular system of parameters of R . The ring R is called a regular ring if R_P is regular local ring, for every prime ideal P of R .

Proposition 3.1. *Let (R, m) be a regular local ring and M be a finitely generated R -module. If $\mathbf{I}(M)$ contains a part of a regular system of parameters, then exactly one of the following holds:*

- (1) $M \cong R/\mathbf{I}(M) \oplus R^{n-1}$,
- (2) $M \cong R^n / \langle (a_1, \dots, a_n)^t \rangle$, for some $n \in \mathbb{N}$ and $a_1, \dots, a_n \in R$.

Proof. Let $\mu(M) = n$ and

$$0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0$$

be an exact sequence. Let $A = (a_{ij})_{n \times m}$ be the matrix presentation of this sequence. (Note that since every regular local ring is Noetherian, hence $\ker(\varphi)$ is finitely generated, and so A has finitely many columns). Since $\mu(M) = n$, it is easily seen that $a_{ij} \in m$, for all i, j . We have $\text{Fitt}_i(M) \subseteq m^{n-i}$. Since $I(M)$ contains a part of a system of parameters, so by [5, Theorem 9.1.1], $I(M) \not\subseteq m^2$. Thus $I(M) = \text{Fitt}_{n-1}(M)$. This means that M is a module of type (F_{n-1}) . Since $I(M) \not\subseteq m^2$, hence there exist some i, j , $1 \leq i \leq n$ and $1 \leq j \leq m$ such that $a_{ij} \notin m^2$. Because every regular local ring is a UFD ([3, Theorem 19.19]), hence a_{ij} is an irreducible element. It is easily seen that every irreducible element in a UFD is prime. So by Theorem 2.7, $M \cong R/I(M) \oplus R^{n-1}$ or $M \cong R^n / \langle (a_1, \dots, a_n)^t \rangle$, for some $n \in \mathbb{N}$ and $a_i \in R$, $1 \leq i \leq n$. \square

Corollary 3.2. *Let (R, m) be a regular local ring and M be a finitely generated R -module. If $I(M) = m$, then*

- (1) $M \cong R/m \oplus R^{n-1}$, if M is not torsionfree,
- (2) $M \cong R^n / \langle (a_{1l}, \dots, a_{nl})^t \rangle$, if M is torsionfree.

Proof. Since m is generated by a regular system of parameters, so by Proposition 3.1, we are done. \square

Now, we generalize Corollary 3.2, to global case.

Theorem 3.3. *Let Q be a maximal ideal of a regular ring R . Let M be a finitely generated R -module such that the R_Q -module M_Q is not torsionfree. Then $I(M) = Q$ if and only if $M \cong P \oplus R/Q$, where P is a projective R -module.*

Proof. Let $I(M) = Q \in \text{Max}(R)$. Then $I(M_Q) = QR_Q$ and $I(M_q) = R_q$ for every maximal ideal $Q \neq q$. By [2, Lemma 1], for every $Q \neq q \in \text{Max}(R)$, there exists some positive integer m such that $M_q \cong R_q^m$. Since M_Q is not a torsionfree R_Q -module, Corollary 3.2 yields that $M_Q \cong R_Q^n \oplus R_Q/QR_Q$, for some positive integer n . So $(M/\mathbf{T}(M))_q$ is free for every maximal ideal q of R . Therefore $M/\mathbf{T}(M)$ is a projective R -module. On the other hand, $\mathbf{T}(M_Q) \cong R_Q/QR_Q$. Thus $\mathbf{T}(M_Q)$ is a simple R_Q -module. Put $\mathcal{A} = \{\text{ann}_R(y) : \mathbf{T}(M_Q) = \langle \frac{y}{1} \rangle\}$. Let $\mathbf{T}(M_Q) = \langle \frac{x}{1} \rangle$ such that $\text{ann}_R(x)$ is maximal in \mathcal{A} . It is easily seen that $\text{ann}_R(x) = Q$. Define $f : R/Q \longrightarrow \mathbf{T}(M)$; $f(r+Q) = rx$. It is clear that f_q is an isomorphism

for every maximal ideal q of R . Hence $\mathbf{T}(M) \cong R/Q$. Therefore $M \cong M/\mathbf{T}(M) \oplus \mathbf{T}(M) \cong P \oplus R/Q$, for some projective R -module P . Conversely, let $M \cong P \oplus R/Q$, for some projective R -module P and maximal ideal Q . Then it is clear that $I(M) = Q$. \square

Corollary 3.4. *Let R be a Noetherian regular ring and Q be a maximal ideal of R . Let M be a finitely generated R -module with $I(M) = Q$. Then*

- (1) *If the R_Q -module M_Q is torsionfree, then $pd_R(M) = 1$,*
- (2) *If the R_Q -module M_Q is not torsionfree, then $pd_R(M) = gldim(R_Q)$.*

Proof. Let $Q \neq q$ be a maximal ideal of R . So $I(M_q) = I(M)_q = R_q$. Thus by [2, Lemma 1], M_q is a free R_q -module. Also we have $I(M_Q) = QR_Q$. If the R_Q -module M_Q is torsionfree, then by Corollary 3.2, $M_Q \cong R_Q^n / \langle (a_1, \dots, a_n)^t \rangle$, for some $a_1, \dots, a_n \in R_Q$. So $pd_R(M) = \sup_q pd(M_q) = 1$.

Next, assume that the R_Q -module M_Q is not torsionfree. So by Corollary 3.2, $M_Q \cong R_Q/QR_Q \oplus R_Q^{n-1}$. Hence $pd_R(M) = \sup_q pd(M_q) = gldim(R_Q)$. \square

Acknowledgement

The authors would like to thank the referee(s) for their useful comments and suggestions.

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