Categories and General Algebraic Structures with Applications

Volume 1, Number 1, December 2013
ISSN Print: 2345-5853 Online: 2345-5861

Shahid Beheshti University
http://www.cgasa.ir

Categories and
General Algebraic Structures
with Applications
Published by: Shahid Beheshti University

Volume 1, Number 1, December 2013, 11-26
ISSN Print: 2345-5853 Online: 2345-5861

Concerning the frame of minimal prime ideals of pointfree function rings

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Abstract. Let L be a completely regular frame and $\mathcal{R}L$ be the ring of continuous real-valued functions on L. We study the frame $\mathfrak{O}(\mathrm{Min}(\mathcal{R}L))$ of minimal prime ideals of $\mathcal{R}L$ in relation to βL. For $I \in \beta L$, denote by $\boldsymbol{O}^I$ the ideal $\{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \in I\}$ of $\mathcal{R}L$. We show that sending I to the set of minimal prime ideals not containing $\boldsymbol{O}^I$ produces a $*$-dense one-one frame homomorphism $\beta L \to \mathfrak{O}(\mathrm{Min}(\mathcal{R}L))$ which is an isomorphism if and only if L is basically disconnected.

1 Introduction

The study of the space of minimal prime ideals of a commutative ring was initiated by Henriksen and Jerison [13]. In that article they relate the space $\mathrm{Min}(C(X))$ to βX by constructing a continuous function $\mathrm{Min}(C(X)) \to \beta X$ which maps no proper closed subset of $\mathrm{Min}(C(X))$

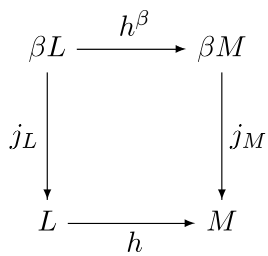
Keywords: frame, ring of real-valued continuous functions on a frame, minimal prime ideal, basically disconnected.
Subject Classification[2000]: 06D22, 54E17, 18A40.
The author acknowledges financial assistance from the National Research Foundation of South Africa.

onto βX, and is a homeomorphism precisely when X is basically connected. Our intent in this article is to study the frame $\mathfrak{O}(\mathrm{Min}(\mathcal{R}L))$ in relation to βL and investigate if there are results that parallel the spatial ones we have just mentioned.

We define a map $\tau_L \colon \beta L \to \mathfrak{O}(\mathrm{Min}(\mathcal{R}L))$ by sending an element of βL to the set of all minimal prime ideals of $\mathcal{R}L$ which do not contain the ideal $\{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \in I\}$ of $\mathcal{R}L$. This turns out to be a one-one frame homomorphism (Proposition 3.1), which is an isomorphism if and only if L is basically disconnected (Proposition 3.2). This accords with the spatial result of Henriksen and Jerison because a topological space is basically disconnected precisely if its frame of open sets is basically disconnected.

A frame homomorphism is called $*$-*dense* if whenever its right adjoint sends an element to the bottom, then that element is the bottom of the codomain of the homomorphism. This notion generalises the property of a continuous map sending no proper closed subset of its domain onto its codomain. We show in Proposition 3.3 that τ_L is $*$-dense, so that, once again, we have a result which is in agreement with its spatial counterpart.

Every frame homomorphism $h \colon L \to M$ between completely regular frames has a *Stone extension* $h^\beta \colon \beta L \to \beta M$, which is a unique frame homomorphism making the square below commute.



For those h for which the ring homomorphism $\mathcal{R}h \colon \mathcal{R}L \to \mathcal{R}M$ contracts minimal prime ideals to minimal prime ideals (for instance whenever L is a P-frame), we construct a frame homomorphism $\bar{h} \colon \mathfrak{O}(\mathrm{Min}(\mathcal{R}L)) \to$

$\mathfrak{O}(\mathrm{Min}(\mathcal{R}M))$ which makes the square

$$\begin{array}{ccc}
\beta L & \xrightarrow{\quad h^{\beta} \quad} & \beta M \\
\Big\downarrow{\scriptstyle \tau_L} & & \Big\downarrow{\scriptstyle \tau_M} \\
\mathfrak{O}(\mathrm{Min}(\mathcal{R}L)) & \xrightarrow{\quad \bar{h} \quad} & \mathfrak{O}(\mathrm{Min}(\mathcal{R}M))
\end{array}$$

commute. If L is basically disconnected, then $\bar{h}$ is unique with this property.

2 Preliminaries

All our frames are completely regular, and our main reference for frames is [14]. For a detailed discussion on the ring of continuous real-valued functions on a frame, the reader should also consult [1] and [2]. We denote the right adjoint of a homomorphism $h\colon L \to M$ by h_*. A homomorphism is called *dense* if it maps only the bottom element to the bottom element; and it is *codense* if the top is the only element it sends to the top. An element p of a frame is called a *point* if $p \neq 1$ and $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. We denote by $\mathrm{Pt}(L)$ the set of all points of L. The points of a regular frame are precisely those elements which are maximal strictly below the top. A *complemented* element in a frame is an element which joins its pseudocomplement at the top.

As in [2], we denote by $\mathcal{R}L$ the ring of all real-valued continuous functions on L. The reader will recall that the underlying set of this ring is the set of all frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to L$, where $\mathfrak{L}(\mathbb{R})$ denotes the frame of reals. A *cozero element* of L is an element of the form $\varphi\big((-,0) \vee (0,-)\big)$, for some $\varphi \in \mathcal{R}L$. An element a of L is a cozero element if and only if there is a sequence (a_n) in L such that $a_n \prec\!\prec a$ for each n and $a = \bigvee a_n$. The set of all cozero elements of L is called the *cozero part* of L and is denoted by $\mathrm{Coz}\, L$. It is a sub-σ-frame of L which generates L if L is completely regular. General properties of cozero elements and cozero parts of frames can be found in [3]. A homomorphism $h\colon L \to M$ is *coz-onto* if for every $d \in \operatorname{Coz} M$ there is a $c \in \operatorname{Coz} L$ with $h(c) = d$. As usual, we denote by βL the Stone-Čech compactification of L, which we take to be the frame of regular ideals of $\operatorname{Coz} L$. For our purposes this is more convenient than viewing βL as the frame of completely regular ideals of L. The right adjoint of the join map $j_L\colon \beta L \to L$ will be denoted by r_L. Because of the way have chosen to view βL, the right adjoint r_L is given by $r_L(a) = \{c \in \operatorname{Coz} L \mid c \prec\!\!\prec a\}$. For each $I \in \beta L$, the ideals $\boldsymbol{O}^I$ and $\boldsymbol{M}^I$ of $\mathcal{R}L$ are defined as follows:

$$\boldsymbol{O}^I = \{\alpha \in \mathcal{R}L \mid r_L(\operatorname{coz}\alpha) \prec I\} \text{ and } \boldsymbol{M}^I = \{\alpha \in \mathcal{R}L \mid r_L(\operatorname{coz}\alpha) \leq I\}.$$

Since for any $I, J \in \beta L$, $I \prec J$ implies $\bigvee I \in J$, it follows that

$$\boldsymbol{O}^I = \{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \in I\}.$$

For any $a \in L$ we abbreviate $\boldsymbol{M}^{\,r_L(a)}$ as $\boldsymbol{M}_a$, and remark that

$$\boldsymbol{M}_a = \{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \leq a\}.$$

It is shown in [6, Lemma 3.1] that, for any $\alpha \in \mathcal{R}L$,

$$\operatorname{Ann}(\alpha) = \boldsymbol{M}_{(\operatorname{coz}\alpha)^*} \quad \text{and} \quad \operatorname{Ann}^2(\alpha) = \boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}}.$$

Furthermore, the annihilator ideals of $\mathcal{R}L$ are exactly the ideals $\boldsymbol{M}_{a^*}$, for $a \in L$. The maximal ideals of $\mathcal{R}L$ are precisely the ideals $\boldsymbol{M}^I$, for $I \in \operatorname{Pt}(\beta L)$; and for any prime ideal P of $\mathcal{R}L$, there is a unique $I \in \operatorname{Pt}(\beta L)$ such that $\boldsymbol{O}^I \subseteq P \subseteq \boldsymbol{M}^I$. See [5] for the proofs of these assertions.

3 The main results

Let us recall what the frame $\mathfrak{O}(\operatorname{Min}(\mathcal{R}L))$ looks like. For any ideal Q in $\mathcal{R}L$, let $\mathcal{U}(Q)$ be the set

$$\mathcal{U}(Q) = \{P \in \operatorname{Min}(\mathcal{R}L) \mid P \not\supseteq Q\}.$$

For the principal ideal $\langle\alpha\rangle$, we abbreviate $\mathcal{U}(\langle\alpha\rangle)$ as $\mathcal{U}(\alpha)$. Then

$$\mathfrak{O}(\mathrm{Min}(\mathcal{R}L)) = \{\mathcal{U}(Q) \mid Q \text{ is an ideal of } \mathcal{R}L\},$$

and the set $\{\mathcal{U}(\alpha) \mid \alpha \in \mathcal{R}L\}$ is a base for this frame consisting of complemented elements, thus making the frame zero-dimensional, and hence completely regular. We shall denote the bottom of this frame by $\perp$, which of course is the empty set, and its top by $\top$. An ideal Q of $\mathcal{R}L$ is called a *z-ideal* if for any $\alpha, \gamma \in \mathcal{R}L$, $\operatorname{coz}\alpha = \operatorname{coz}\gamma$ and $\alpha \in Q$ imply $\gamma \in Q$. The equal sign can be replaced with $\leq$. Minimal prime ideals are z-ideals. Define the map

$$\tau_L\colon \beta L \to \mathfrak{O}(\mathrm{Min}(\mathcal{R}L)) \quad \text{by} \quad \tau_L(I) = \mathcal{U}(\boldsymbol{O}^I).$$

Proposition 3.1. *For any completely regular frame L, the map τ_L is a one-one frame homomorphism.*

Proof. Clearly, τ_L preserves the bottom and the top. It also preserves binary meets because, for any $I, J \in \beta L$, $\boldsymbol{O}^{I\wedge J} = \boldsymbol{O}^I \cap \boldsymbol{O}^J$. Let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a collection of elements of βL, and, for brevity, write $I = \bigvee_\lambda I_\lambda$. We show that $\tau_L\big(I\big) \subseteq \bigcup_\lambda \tau_L(I_\lambda)$, which will prove that τ_L preserves joins since it preserves order. Let P be in $\tau_L(I)$. Then $\boldsymbol{O}^I \nsubseteq P$, and so there is an $\alpha \in \boldsymbol{O}^I$ such that $\alpha \notin P$. By the way joins are calculated in βL, there are indices $\lambda_1, \dots, \lambda_n$ in Λ, and elements $c_i \in I_{\lambda_i}$, for $i = 1, \dots, n$, such that

$$\operatorname{coz}\alpha = c_1 \vee \cdots \vee c_m.$$

For each i, take a positive $\gamma_i \in \mathcal{R}L$ such that $c_i = \operatorname{coz}(\gamma_i)$. Suppose, by way of contradiction, that $P \notin \bigcup_\lambda \tau_L(I_\lambda)$. Then $\boldsymbol{O}^{I_\lambda} \subseteq P$ for every $\lambda \in \Lambda$. In particular, $\boldsymbol{O}^{I_{\lambda_i}} \subseteq P$ for each $i = 1, \dots, n$, which implies $\gamma_i \in P$ for each i, and hence $\gamma_1 + \cdots + \gamma_n \in P$. Since $\operatorname{coz}\alpha = \operatorname{coz}(\gamma_1 + \cdots + \gamma_n)$ and P is a z-ideal, we have that $\alpha \in P$, and thus we have reached a contradiction. Therefore τ_L is a frame homomorphism.

Since the frames βL and $\mathfrak{O}(\mathrm{Min}(\mathcal{R}L))$ are regular, to prove that τ_L is one-one it suffices to show that τ_L is codense. Consider therefore any $I \in \beta L$ with $\tau_L(I) = \top$. This implies $\mathcal{U}(\boldsymbol{O}^I) = \top$, so that $\boldsymbol{O}^I \nsubseteq P$, for any minimal prime ideal P of $\mathcal{R}L$. Suppose, for contradiction, that $I \neq 1_{\beta L}$. Since βL has enough points, take a point $J \in \mathrm{Pt}(\beta L)$ with $I \leq J$. The maximal ideal $\boldsymbol{M}^J$ contains a minimal prime ideal, say P. Then $\boldsymbol{O}^J \subseteq P$. Since $I \leq J$, we have $\boldsymbol{O}^I \subseteq \boldsymbol{O}^J \subseteq P$; and hence a contradiction. Therefore $I = 1_{\beta L}$, as required. □

Recall that a frame L is *basically disconnected* if $c^* \vee c^{**} = 1$ for every $c \in \operatorname{Coz} L$. Observe that if $a \in L$ is complemented, then $\boldsymbol{O}^{r_L(a)} = \boldsymbol{M}_a$. This is so because if $\alpha \in \boldsymbol{M}_a$ then $\operatorname{coz}\alpha \leq a \prec\!\prec a$, so that $\operatorname{coz}\alpha \in r_L(a)$, hence $\alpha \in \boldsymbol{O}^{r_L(a)}$. We will need some results from elsewhere.

For a commutative ring A with identity, let $\mathrm{Max}(A)$ denote the space of maximal ideals of A with the hull-kernel topology. Recall that the topology of $\mathrm{Max}(A)$ is precisely the frame

$$\mathfrak{O}(\mathrm{Max}(A)) = \{\mathcal{M}(Q) \mid Q \text{ is an ideal of } A\},$$

where, for any ideal Q of A,

$$\mathcal{M}(Q) = \{M \in \mathrm{Max}(A) \mid M \not\supseteq Q\}$$

As before we write $\mathcal{M}(a)$ for $\mathcal{M}(\langle a \rangle)$. Scott Woodward proved in his PhD thesis [15] that if A is an f-ring with zero Jacobson radical, then a subset of $\mathrm{Max}(A)$ is clopen precisely if it is of the form $\mathcal{M}(e)$, for some idempotent $e \in A$.

It can be deduced from results in [4] that, for any completely regular frame L,

$$\mathfrak{O}(\mathrm{Max}(\mathcal{R}L)) \cong \beta L,$$

in perfect analogy with the spatial result that $\mathrm{Max}(C(X))$ is homeomorphic to βX, for any Tychonoff space X. For each ideal Q of $\mathcal{R}L$, denote by I_Q the element of βL given by

$$I_Q = \bigvee\{r_L(\operatorname{coz}\alpha) \mid \alpha \in Q\}.$$

A careful analysis reveals that the map

$$\varrho_L\colon \mathfrak{O}(\mathrm{Max}(\mathcal{R}L)) \to \beta L \quad \text{defined by} \quad \varrho_L(\mathcal{M}(Q)) = I_Q$$

is well defined, and is a frame isomorphism. We shall demonstrate only that it is well defined. For this we need only show that if P and Q are ideals of $\mathcal{R}L$ with $\mathcal{M}(P) = \mathcal{M}(Q)$, then $I_P = I_Q$. Observe that, for any $J \in \mathrm{Pt}(\beta L)$,

$$\begin{aligned} Q \subseteq \boldsymbol{M}^J \quad &\Longleftrightarrow \quad r_L(\operatorname{coz}\alpha) \leq J \text{ for every } \alpha \in Q \\ &\Longleftrightarrow \quad I_Q \leq J. \end{aligned}$$

Since I_Q is the meet of points of βL above it, and since $\mathcal{M}(P) = \mathcal{M}(Q)$ if and only if P and Q are contained in exactly the same maximal ideals, it follows that $I_P = I_Q$.

We remind the reader that an ideal I of a commutative ring is called a *d-ideal* if, for every $a \in I$, $\mathrm{Ann}^2(a) \subseteq I$. Minimal prime ideals are d-ideals.

Proposition 3.2. *τ_L is an isomorphism iff L is basically disconnected.*

Proof. ($\Rightarrow$) Assume τ_L is an isomorphism. By [7, Proposition 3.3], it suffices to show that the annihilator of every element of $\mathcal{R}L$ is a principal ideal generated by an idempotent. By the current hypothesis, the composite

$$\mathfrak{O}(\mathrm{Max}(\mathcal{R}L)) \xrightarrow{\ \varrho_L\ } \beta L \xrightarrow{\ \tau_L\ } \mathfrak{O}(\mathrm{Min}(\mathcal{R}L))$$

is an isomorphism. Let $\alpha \in \mathcal{R}L$. Since $\mathcal{U}(\alpha)$ is complemented, the element of $\mathfrak{O}(\mathrm{Max}(\mathcal{R}L))$ mapped to it by the isomorphism $\tau_L \cdot \varrho_L$ is complemented, and so, by the result of Woodward cited above, there is an idempotent $\eta \in \mathcal{R}L$ such that $\tau_L\varrho_L(\mathcal{M}(\eta)) = \mathcal{U}(\alpha)$. It is clear that $I_{\langle\eta\rangle} = r_L(\mathrm{coz}\,\eta)$, so that $\mathcal{U}(\alpha) = \mathcal{U}(\boldsymbol{O}^{r_L(\mathrm{coz}\,\eta)})$. Since $\mathrm{coz}\,\eta$ is complemented (as η is an idempotent), $\eta \in r_L(\mathrm{coz}\,\eta)$, and hence, for any $P \in \mathrm{Min}(\mathcal{R}L)$,

$$\eta \notin P \iff \boldsymbol{O}^{r_L(\mathrm{coz}\,\eta)} \nsubseteq P.$$

Thus, $\mathcal{U}(\alpha) = \mathcal{U}(\eta)$, and hence, by [13, Theorem 2.7],

$$\mathrm{Ann}(\alpha) = \bigcap \mathcal{U}(\alpha) = \bigcap \mathcal{U}(\eta) = \mathrm{Ann}(\eta) = \langle \mathbf{1} - \eta \rangle.$$

Since $\mathbf{1} - \eta$ is an idempotent, we are done.

($\Leftarrow$) We need only show that τ_L is surjective. Because the elements $\mathcal{U}(\alpha)$ form a base for $\mathfrak{O}(\mathrm{Min}(\mathcal{R}L))$, we shall be done if we show that each such element has something mapped to it. Now, since minimal prime ideals are d-ideals, for any $\alpha \in \mathcal{R}L$ and minimal prime ideal P of $\mathcal{R}L$, we have

$$\alpha \notin P \iff \mathrm{Ann}^2(\alpha) \nsubseteq P,$$

so that, in light of $\mathrm{Ann}^2(\alpha) = \boldsymbol{M}_{(\mathrm{coz}\,\alpha)^{**}} = \boldsymbol{O}^{r_L((\mathrm{coz}\,\alpha)^{**})}$, as $(\mathrm{coz}\,\alpha)^{**}$ is complemented since L is basically connected,

$$\alpha \notin P \iff \boldsymbol{O}^{r_L((\operatorname{coz}\alpha)^{**})} \nsubseteq P.$$

This implies $\tau_L(\boldsymbol{O}^{r_L((\operatorname{coz}\alpha)^{**})}) = \mathcal{U}(\alpha)$, which shows that τ_L is onto, and is therefore an isomorphism. □

Recall that a homomorphism $h\colon L \to M$ is said to be $*$-*dense* [12] if, for any $b \in M$, $h_*(b) = 0$ implies $b = 0$. This captures, in a slightly more general form, the notion of a surjective continuous function $f\colon X \to Y$ being irreducible, in the sense that $f[K] = Y$ for any closed $K \subseteq X$ implies $K = X$.

Proposition 3.3. *τ_L is $*$-dense.*

Proof. We first calculate the right adjoint of τ_L. With the notation as above, note that, for any ideal P of $\mathcal{R}L$,

$$I_P = \bigcup\{r_L(\operatorname{coz}\alpha) \mid \alpha \in P\}$$

because the join defining I_P is directed. We show that $\tau_L(I_P) \subseteq \mathcal{U}(P)$. To start, observe that $\boldsymbol{O}^{I_P} \subseteq P$. Indeed, let $\alpha \in \boldsymbol{O}^{I_P}$. Then $\operatorname{coz}\alpha \in I_P$, implying $\operatorname{coz}\alpha \prec\!\prec \operatorname{coz}\beta$ for some $\beta \in P$. By [5, Lemma 4.4], this implies α is a multiple of β, whence $\alpha \in P$. Therefore

$$\tau_L(I_P) = \mathcal{U}(\boldsymbol{O}^{I_P}) \subseteq \mathcal{U}(P).$$

Now, given any ideal Q of $\mathcal{R}L$, let $\bar{Q}$ be the subset of $\mathcal{R}L$ defined by

$$\bar{Q} = \bigcup\{T \mid T \text{ is an ideal of } \mathcal{R}L \text{ with } \mathcal{U}(T) = \mathcal{U}(Q)\}.$$

The collection whose union is computed is directed because $\mathcal{U}(T_1) = \mathcal{U}(T_2) = \mathcal{U}(Q)$ implies $\mathcal{U}(T_1 + T_2) = \mathcal{U}(Q)$. Thus, $\bar{Q}$ is an ideal, and, in fact, the largest ideal of $\mathcal{R}L$ with $\mathcal{U}(\bar{Q}) = \mathcal{U}(Q)$. We claim that

$$(\tau_L)_*(\mathcal{U}(Q)) = I_{\bar{Q}}.$$

As observed above, $\tau_L(I_{\bar{Q}}) \subseteq \mathcal{U}(\bar{Q}) = \mathcal{U}(Q)$. Consider any $J \in \beta L$ with $\tau_L(J) \subseteq \mathcal{U}(Q)$. Then $\mathcal{U}(\boldsymbol{O}^J) \subseteq \mathcal{U}(Q)$, which implies

$$\boldsymbol{O}^J \subseteq \boldsymbol{O}^J + Q \subseteq \bar{Q}.$$

Now let $a \in J$ and take a $\gamma \in \mathcal{R}L$ such that $a \prec\!\prec \operatorname{coz}\gamma \in J$. Then $\gamma \in \boldsymbol{O}^J \subseteq \bar{Q}$, which shows that $a \in I_{\bar{Q}}$. Therefore $J \subseteq I_{\bar{Q}}$, and hence $(\tau_L)_*(\mathcal{U}(Q)) = I_{\bar{Q}}$, as claimed.

Suppose now that $\mathcal{U}(Q)$ is such that $(\tau_L)_*(\mathcal{U}(Q)) = 0_{\beta L}$. Then $I_{\bar{Q}} = 0_{\beta L}$, which, by complete regularity, implies $\bar{Q} = \{\mathbf{0}\}$, and hence $\mathcal{U}(Q) = \mathcal{U}(\bar{Q}) = \bot$. So τ_L is $*$-dense. □

In the introduction we recalled the Stone extension $h^\beta\colon \beta L \to \beta M$ of a frame homomorphism $h\colon L \to M$. We remind the reader that, because of the way we view βL, the map h^β is given by

$$h^\beta(I) = \{c \in \operatorname{Coz} M \mid c \le h(d) \text{ for some } d \in I\}.$$

In light of the above, we have the wedge

$$\begin{array}{ccc} \beta L & \xrightarrow{\quad h^{\beta} \quad} & \beta M \\ {\scriptstyle \tau_L}\big\downarrow & & \big\downarrow{\scriptstyle \tau_M} \\ \mathfrak{O}(\mathrm{Min}(\mathcal{R}L)) & & \mathfrak{O}(\mathrm{Min}(\mathcal{R}M)) \end{array}$$

which we would like to complete into a commutative square by filling in a homomorphism, say $\bar{h}\colon \mathfrak{O}(\mathrm{Min}(\mathcal{R}L)) \to \mathfrak{O}(\mathrm{Min}(\mathcal{R}M))$, induced by h. We shall need to restrict the map h by requiring that the inverse image of any minimal prime ideal of $\mathcal{R}M$ under the ring homomorphism $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}M$ be minimal prime. This might sound too stringent, but observe that any homomorphism out of a P-frame has this property because L is a P-frame if and only if every prime ideal of $\mathcal{R}L$ is minimal prime [5, Proposition 4.9].

Let us introduce the following notation. Given a homomorphism $h\colon L \to M$ and an ideal Q of $\mathcal{R}L$, we set

$$Q_{(h)} = \{\gamma \in \mathcal{R}M \mid \operatorname{coz}\gamma \leq h(\operatorname{coz}\alpha) \text{ for some } \alpha \in Q\}.$$

A routine calculation, using properties of the cozero map, shows that $Q_{(h)}$ is an ideal of $\mathcal{R}M$, which is proper if and only if h is *coz-codense*, meaning that the only cozero it takes to the top is the top. Saying "$(\mathcal{R}h)^{-1}[P]$ is a minimal prime ideal for very minimal prime ideal P of $\mathcal{R}M$" is quite a mouthful, so we shall say h is *balanced* if it has this property.

Lemma 3.4. *Let $h\colon L \to M$ be a balanced homomorphism. For any ideal Q of $\mathcal{R}L$, and any $T \in \mathrm{Min}(\mathcal{R}M)$, $Q_{(h)} \nsubseteq T$ iff $Q \nsubseteq (\mathcal{R}h)^{-1}[T]$.*

Proof. Suppose $Q_{(h)} \nsubseteq T$, and take $\beta \in Q_{(h)}$ with $\beta \notin T$. Pick $\alpha \in Q$ such that

$$\operatorname{coz}\beta \leq h(\operatorname{coz}\alpha) = \operatorname{coz}(\mathcal{R}h(\alpha)).$$

Since T is a z-ideal and $\beta \notin T$, we must have $\mathcal{R}h(\alpha) \notin T$, whence $\alpha \notin (\mathcal{R}h)^{-1}[T]$. Therefore $Q \nsubseteq (\mathcal{R}h)^{-1}[T]$. Conversely, if γ is in Q but not in $(\mathcal{R}h)^{-1}[T]$, then $\mathcal{R}h(\gamma)$ is in $Q_{(h)}$ but not in T, showing that $Q_{(h)} \nsubseteq T$. □

In what follows we use subscripts on $\mathcal{U}$ to indicate the frame with reference to which the collection of minimal prime ideals is being contemplated. Let $h\colon L \to M$ be a balanced homomorphism. Define

$$\bar{h}\colon \mathfrak{O}(\mathrm{Min}(\mathcal{R}L)) \to \mathfrak{O}(\mathrm{Min}(\mathcal{R}M)) \quad \text{by} \quad \bar{h}(\mathcal{U}_L(Q)) = \mathcal{U}_M(Q_{(h)}).$$

Since $\mathcal{U}_L(Q)$ is not uniquely determined by Q, we must check that $\bar{h}$ is a well-defined function. Suppose $\mathcal{U}_L(Q) = \mathcal{U}_L(R)$ for some ideals Q and R in $\mathcal{R}L$. Let $T \in \mathcal{U}_M(Q_{(h)})$. Then $Q_{(h)} \nsubseteq T$, so that, by the lemma above, $Q \nsubseteq (\mathcal{R}h)^{-1}[T]$, whence $R \nsubseteq (\mathcal{R}h)^{-1}[T]$, thence $R_{(h)} \nsubseteq T$. Therefore $\mathcal{U}_M(Q_{(h)}) \subseteq \mathcal{U}_M(R_{(h)})$, and hence equality by symmetry.

Proposition 3.5. *Let $h\colon L \to M$ be a balanced homomorphism. The map $\bar{h}$ is a frame homomorphism making the square*

$$\begin{array}{ccc} \beta L & \xrightarrow{\;h^{\beta}\;} & \beta M \\ \downarrow{\scriptstyle \tau_L} & & \downarrow{\scriptstyle \tau_M} \\ \mathfrak{O}(\mathrm{Min}(\mathcal{R}L)) & \xrightarrow{\;\bar{h}\;} & \mathfrak{O}(\mathrm{Min}(\mathcal{R}M)) \end{array}$$

commute. If L is basically disconnected, then $\bar{h}$ is unique with this property.

Proof. It is immediate that $\bar{h}$ preserves the bottom and the top. An easy application of Lemma 3.4 shows that $\bar{h}$ preserves order. We show that $\bar{h}$ preserves binary meets. Consider any two ideals P and Q in $\mathcal{R}L$. It suffices to show that

$$\bar{h}\Big(\mathcal{U}_L(P)\Big) \cap \bar{h}\Big(\mathcal{U}_L(Q)\Big) \subseteq \bar{h}\Big(\mathcal{U}_L(P) \cap \mathcal{U}_L(Q)\Big) = \bar{h}\Big(\mathcal{U}_L(PQ)\Big).$$

Let $T \in \bar{h}\Big(\mathcal{U}_L(P)\Big) \cap \bar{h}\Big(\mathcal{U}_L(Q)\Big)$. Then $P_{(h)} \nsubseteq T$ and $Q_{(h)} \nsubseteq T$, which, by Lemma 3.4, implies $P \nsubseteq (\mathcal{R}h)^{-1}[T]$ and $Q \nsubseteq (\mathcal{R}h)^{-1}[T]$, so that $PQ \nsubseteq (\mathcal{R}h)^{-1}[T]$, since $(\mathcal{R}h)^{-1}[T]$ is a prime ideal. Consequently, $(PQ)_{(h)} \nsubseteq T$, and hence $T \in \bar{h}\big(\mathcal{U}_L(PQ)\big)$. Therefore $\bar{h}$ preserves binary meets.

Next, let $\{\mathcal{U}_L(Q_i) \mid i \in I\}$ be a collection of elements of $\mathfrak{O}(\mathrm{Min}(\mathcal{R}L))$. We aim to show that $\bar{h}\Big(\bigvee_i \mathcal{U}_L(Q_i)\Big) \leq \bigvee_i \bar{h}\Big(\mathcal{U}_L(Q_i)\Big)$. Put $P = \sum Q_i$. Then

$$\bar{h}\Big(\bigvee_i \mathcal{U}_L(Q_i)\Big) = \bar{h}\Big(\bigcup_i \mathcal{U}_L(Q_i)\Big) = \bar{h}\Big(\mathcal{U}_L(P)\Big) = \mathcal{U}_M(P_{(h)}).$$

Let $T \in \mathcal{U}_M(P_{(h)})$. Then, by Lemma 3.4, $\sum Q_i \nsubseteq (\mathcal{R}h)^{-1}[T]$, which implies that there is an index $i_0 \in I$ for which $Q_{i_0} \nsubseteq (\mathcal{R}h)^{-1}[T]$, so that $(Q_{i_0})_{(h)} \nsubseteq T$. Consequently,

$$T \in \mathcal{U}_M((Q_{i_0})_{(h)}) \subseteq \bigcup_i \mathcal{U}_M((Q_i)_{(h)}) = \bigvee_i \bar{h}\Big(\mathcal{U}_L(Q_i)\Big).$$

Therefore $\bar{h}$ is a frame homomorphism.

To show that the square commutes, let $I \in \beta L$. Then

$$\bar{h}\tau_L(I) = \bar{h}(\mathcal{U}_L(\boldsymbol{O}^I)) = \mathcal{U}_M(\boldsymbol{O}^I_{(h)}),$$

and

$$\tau_M h^\beta(I) = \mathcal{U}_M(\boldsymbol{O}^{h^\beta(I)}).$$

We finish the proof by showing that $\boldsymbol{O}^I_{(h)} = \boldsymbol{O}^{h^\beta(I)}$. Let $\gamma \in \boldsymbol{O}^I_{(h)}$. Then $\operatorname{coz}\gamma \leq h(\operatorname{coz}\alpha)$ for some $\alpha \in \boldsymbol{O}^I$. But $\alpha \in \boldsymbol{O}^I$ implies $\operatorname{coz}\alpha \in I$, so that $\operatorname{coz}\gamma \in h^\beta(I)$, whence $\gamma \in \boldsymbol{O}^{h^\beta(I)}$. Therefore $\boldsymbol{O}^I_{(h)} \subseteq \boldsymbol{O}^{h^\beta(I)}$. On the other hand, let $\sigma \in \boldsymbol{O}^{h^\beta(I)}$. Then $\operatorname{coz}\sigma \in h^\beta(I)$, which implies $\operatorname{coz}\sigma \leq h(\operatorname{coz}\mu)$ for some μ with $\operatorname{coz}\mu \in I$. Thus $\mu \in \boldsymbol{O}^I$, and therefore $\sigma \in \boldsymbol{O}^I_{(h)}$.

Now suppose L is basically disconnected and that $g\colon \mathfrak{O}(\operatorname{Min}(\mathcal{R}L)) \to \mathfrak{O}(\operatorname{Min}(\mathcal{R}M))$ satisfies $g \cdot \tau_L = \tau_M \cdot h^\beta$. Then $g \cdot \tau_L = \bar{h} \cdot \tau_L$, and hence $g = \bar{h}$ because τ_L is an isomorphism by Proposition 3.2. □

Remark 1. In [8] it is shown that, for a surjective frame homomorphism $h\colon L \to M$, the ring homomorphism $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}M$ contracts maximal ideals to maximal ideals if and only if, for every $c \in \operatorname{Coz} L$ and $d \in \operatorname{Coz} M$ with $h(c) \vee d = 1$, there is a $u \in \operatorname{Coz} L$ such that $u \vee c = 1$, and $h(u) \leq d$. We have not determined if there is such an element-wise characterisation for balanced maps.

4 Concluding observations regarding $\operatorname{Min}(\mathcal{R}L)$

It is shown in [13] that, for any Tychonoff space X, $\operatorname{Min}(C(X))$ is countably compact, and it is compact and basically disconnected precisely when every open set is dense in some cozero-set. We conclude by demonstrating that the same results hold for frames.

A ring A is said to satisfy the *countable annihilator condition* [13], or is called a *c.a.c. ring*, if for any sequence (a_n) in A, there is an $x \in A$ such that $\operatorname{Ann}(x) = \bigcap_{n=1}^{\infty} \operatorname{Ann}(a_n)$. It is observed in [6] that $\mathcal{R}L$ is a c.a.c. ring. Consequently, in view of [13, Theorem 4.9], we have the following result.

Proposition 4.1. $\mathrm{Min}(\mathcal{R}L)$ *is countably compact for any completely regular frame* L.

Following [10], we say L is *cozero approximated* if, for every $x \in L$, there is an $a \in \mathrm{Coz}\, L$ such that $a^* = x^*$. In spaces this says for every open set $U \subseteq X$, there is a cozero set V of X such that $\overline{U} = \overline{V}$. In [11] a space with this property is called *fraction dense*. Theorem 4.4 of [13] states that if A is an a.c. ring (a weaker form of the c.a.c. property), then $\mathrm{Min}(A)$ is compact and extremally disconnected precisely when for every $B \subseteq A$ there is a $y \in A$ such that $\mathrm{Ann}(B) = \mathrm{Ann}(y)$. Now since annihilator ideals of $\mathcal{R}L$ are precisely the ideals $\boldsymbol{M}_{a^*}$ for $a \in L$, and element-annihilators are exactly the ideals $\boldsymbol{M}_{c^*}$, for $c \in \mathrm{Coz}\, L$, we have the following.

Proposition 4.2. $\mathrm{Min}(\mathcal{R}L)$ *is compact and basically disconnected iff* L *is cozero approximated.*

The ring $\mathcal{R}L$ is an f-ring with *bounded inversion*, which is to say every $\alpha \geq \mathbf{1}$ is invertible. The bounded part of $\mathcal{R}L$ is denoted by $\mathcal{R}^*L$. An easy algebraic calculation shows that $\frac{\alpha}{\mathbf{1}+|\alpha|} \in \mathcal{R}^*L$ for any $\alpha \in \mathcal{R}L$. Since $\alpha = \frac{\alpha}{\mathbf{1}+|\alpha|} \cdot (\mathbf{1} + |\alpha|)$, and $\mathrm{Ann}(\mathbf{1} + |\alpha|)$ is the zero ideal, it follows from [13, Theorem 5.1] that $\mathrm{Min}(\mathcal{R}(\beta L))$ is homeomorphic to $\mathrm{Min}(\mathcal{R}L)$. Consequently, βL *is cozero approximated iff* L *is cozero approximated.*

Remark 2. That βL is cozero approximated if and only if L is cozero approximated can also be deduced from these two results: (i) if $h\colon L \to M$ is dense onto and L is cozero approximated, then so is M. This is straightforward. (ii) If $h\colon L \to M$ is dense coz-onto and M is cozero approximated, then so is L. To see this, use the fact that if $g\colon N \to K$ is a dense frame homomorphism, then $x^* = g_*g(x^*)$ for every $x \in N$ (see [9, Lemma 3.1]).

Bibliography

[1] R.N. Ball and J. Walters-Wayland, *C- and C^*-quotients in pointfree topology*, Dissertationes Mathematicae (Rozprawy Mat.), Vol. **412** (2002), 62pp.

[2] B. Banaschewski, *The real numbers in pointfree topology*, Textos de Matemática Série B, No. 12, Departamento de Matemática da Universidade de Coimbra, 1997.

[3] B. Banaschewski, C. Gilmour, *Pseudocompactness and the cozero part of a frame*, Comment. Math. Univ. Carolin. **37** (1996), 577-587.

[4] B. Banaschewski and M. Sioen, *Ring ideals and the Stone-Čech compactification in pointfree topology*, J. Pure Appl. Algebra **214** (2010), 2159-2164.

[5] T. Dube, *Some ring-theoretic properties of almost P-frames*, Alg. Univ., **60** (2009), 145-162.

[6] T. Dube, *Contracting the socle in rings of continuous functions*, Rend. Sem. Mat. Univ. Padova **123** (2010), 37-53.

[7] T. Dube, *Notes on pointfree disconnectivity with a ring-theoretic slant*, Appl. Categor. Struct. **18** (2010), 55-72.

[8] T. Dube and M. Matlabyane, *Concerning some variants of C-embedding in pointfree topology*, Top. Appl. **158** (2011), 2307-2321.

[9] T. Dube and I. Naidoo, *On openness and surjectivity of lifted frame homomorphisms*, Top. Appl. **157** (2010), 2159-2171.

[10] G. Gruenhage, *Products of cozero complemented spaces*, Houst. J. Math. **32** (2006), 757-773.

[11] A.W. Hager and J. Martínez, *Fraction-dense algebras and spaces*, Canad. J. Math. **45** (1993), 977-996.

[12] A.W. Hager and J. Martínez, *Patch-generated frames and projectable hulls*, Appl. Categor. Struct. **15** (2007), 49-80.

[13] M. Henriksen and M. Jerison, *The space of minimal prime ideals of a commutative ring*, Trans. Amer. Math. Soc. **115** (1965), 110-130.

[14] J. Picado and A. Pultr, Frames and Locales: topology without points, Frontiers in Mathematics, Springer, Basel (2012).

[15] S. Woodward, *On f-rings which are rich in idempotents*, PhD thesis (1992), University of Florida.

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