Categories and General Algebraic Structures with Applications

Volume 1, Number 1, December 2013
ISSN Print: 2345-5853 Online: 2345-5861

Shahid Beheshti University
http://www.cgasa.ir

Categories and
General Algebraic Structures
with Applications
Published by: Shahid Beheshti University
Volume 1, Number 1, December 2013, 103-117
ISSN Print: 2345-5853 Online: 2345-5861
WWW.CGASA.ir

A characterization of a pomonoid S all of its cyclic S-posets are regular injective

Xia Zhang, Wenling Zhang, Ulrich Knauer

Abstract. This work is devoted to give a charcaterization of a pomonoid S such that all cyclic S-posets are regular injective.

1 Introduction and Preliminaries

In this paper, S will be a *pomonoid*, that is, a monoid equipped with a partial order relation $\leq$ which is compatible with the semigroup multiplication in the sense that $s \leq t$ implies $su \leq tu$ and $us \leq ut$ for every $s, t, u \in S$. A poset $(A, \leq)$ together with a mapping $A \times S \to A$ (under which a pair (a, s) maps to an element of A denoted by as) is called a *right S-poset*, denoted by A_S (or simply A), if for any $a, b \in A,\ s, t \in S$,

(1) $a(st) = (as)t$,

(2) $a1 = a$,

Keywords: Promonoid, Regular injectivity, Cyclic S-poset.
Subject Classification[2000]: 06F05, 20M30, 20M50.
Research of the first named author was supported by the SRFDP of Ministry of Education of China (20124407120004), the NNSF of China (11171118), and the Program on International Cooperation and Innovation, Department of Education, Guangdong Province (2012gjhz0007).

(3) $a \leq b,\ s \leq t \Rightarrow as \leq bt.$

A *left S-poset* can be defined similarly. We only consider right S-posets in the paper, and the word "right" will be omitted. *Homomorphisms of S-posets* are order-preserving mappings which also preserve the S-action. An *S-subposet* of an S-poset A is an action-closed subset of A whose partial order is the restriction of the order of A.

A *preorder* on a set A is a reflexive, transitive binary relation on A (see [1]). A preorder $\leq$ on an S-poset A is *compatible* if $x \leq y$ in A then $xs \leq ys$ for any $s \in S$. Similar to [2], we give the notions of an α-chain in a pomonoid S and of a right order congruence on S. Let α be a right compatible preorder on S. For elements $a,\ a' \in S$, an *α-chain* from a to a' is a sequence of the form

$$a \leq a_1\ \alpha\ a'_1 \leq a_2\ \alpha\ a'_2 \leq \cdots \leq a_n\ \alpha\ a'_n \leq a',$$

where each $a_i,\ a'_i \in S$. We write $a \underset{\alpha}{\leq} a'$ if such a sequence exists.

The following lemma is obvious for an α-chain.

Lemma 1.1. *Let* $(S, \leq)$ *be a pomonoid,* α *a right compatible preorder on* S*, and* $a,\ a',\ a'' \in S$*. Then the following statements hold.*

(1) $a \leq a' \Rightarrow a \underset{\alpha}{\leq} a'$,

(2) $a\ \alpha\ a' \Rightarrow a \underset{\alpha}{\leq} a'$,

(3) $a \underset{\alpha}{\leq} a',\ a' \underset{\alpha}{\leq} a'' \Rightarrow a \underset{\alpha}{\leq} a''$.

For a monoid S, a *right congruence* on S is an equivalence relation on S which is right compatible with the multiplication of S.

Definition 1.2. (cf. [2]) Let S be a pomonoid. A *right order congruence* σ on S is a congruence on the S-poset S_S, that is, σ is a right congruence on S, with the property that S/σ can be equipped with a partial order such that S/σ is an S-poset and the canonical mapping $S \to S/\sigma$ is an S-poset homomorphism.

The following corollaries follow immediately.

Corollary 1.3. (*cf.* [2]) *Let S be a pomonoid and σ a right compatible preorder on S. Then the relation θ_σ defined on S by*

$$s\theta_\sigma t \Leftrightarrow s \underset{\sigma}{\leq} t \underset{\sigma}{\leq} s$$

is a right order congruence on S, a suitable order relation on S/θ_σ being

$$[s]_{\theta_\sigma} \leq [t]_{\theta_\sigma} \Leftrightarrow s \underset{\sigma}{\leq} t.$$

Furthermore, if η is any right order congruence on S such that $\sigma \subseteq \eta$, then $\theta_\sigma \subseteq \eta$ as well. θ_σ is called the right order congruence generated by σ.

Corollary 1.4. *Let S be a pomonoid. Then an S-poset A is cyclic if and only if there exists a right compatible preorder σ on S such that $A \cong S/\theta_\sigma$, where θ_σ is the right order congruence generated by σ.*

Proof. Clearly S/θ_σ is a cyclic S-poset. For the converse, if A is a cyclic S-poset, then there exists $a \in A$ such that $A = aS$. Define a binary relation on S by

$$\sigma = \{(s,\ t) \in S \times S \mid as \leq at\}.$$

Obviously σ is a right compatible preorder on S. Moreover, define a map $f:\ aS\ \rightarrow\ S/\theta_\sigma$ by

$$f(as) = [s]_{\theta_\sigma}.$$

It is routine to check that f is an S-poset isomorphism. □

We are going to study the regular injectivity of cyclic S-posets by using similar techniques as in [5]. First recall some basic definitions and lemmas from [4].

An S-poset Q is *regular injective* if and only if for any S-subposet B of an S-poset A, any S-poset homomorphism $f : B \to Q$, there exists an S-poset homomorphism $g : A \to Q$ extending f, i.e., $g\mid_B = f$ (compare for example [3]).

For an S-poset A, an element $\theta \in A$ is said to be a *zero element* if $\theta s = \theta$ for all $s \in S$.

An S-subposet B of an S-poset A is called *strongly convex* if for any $a \in A$, $b \in B$, $a \leq b$ implies that $a \in B$. If for any S-poset A, all of its S-subposets are strongly convex, then we call S *completely strongly convex.*

In the following, S will be a completely strongly convex pomonoid. If K is a non-empty subset of S such that $KS \subseteq K$ then K is called a *right ideal* of S.

Lemma 1.5. ([4]) *Let S be a completely strongly convex pomonoid and Q an S-poset with a zero. Then Q is regular injective if and only if Q is regular injective relative to all embeddings into cyclic S-posets.*

Lemma 1.6. ([4]) *Every regular injective S-poset contains a zero.*

2 Characterization

In this section, we will characterize a pomonoid S all of its cyclic S-posets are regular injective by using right order congruences on S.

Similar to [5], we give the following notations on a pomonoid S. Let K be a right ideal of S, s an element of S, μ a right compatible preorder on S, and θ_μ the right order congruence generated by μ.

Set

$$\overline{K}_{\theta_\mu} = \{[k]_{\theta_\mu} \in S/\theta_\mu | k \in K\}$$

and

$$K(s, \theta_\mu) = \{a \in S \mid [sa]_{\theta_\mu} \in \overline{K}_{\theta_\mu}\}.$$

Obviously $\overline{K}_{\theta_\mu}$ is an S-subposet of the cyclic S-poset S/θ_μ.

By a routine check, we get the following lemma.

Lemma 2.1. *Let μ, λ be right compatible preorders on S, $\theta_\mu, \theta_\lambda$ the right order congruences generated by μ, λ, respectively, K a right ideal of S, and $q \in S$. Define a relation $\mathcal{R}(K, \theta_\mu, \theta_\lambda, q)$ on S by*

$$s \; \mathcal{R}(K, \theta_\mu, \theta_\lambda, q) \; t \Leftrightarrow K(t, \theta_\mu) \subseteq K(s, \theta_\mu) \text{ and } (qsa) \underset{\lambda}{\leq} (qta) \text{ for all } a \in K(t, \theta_\mu).$$

Then $\mathcal{R}(K, \theta_\mu, \theta_\lambda, q)$ is a right compatible preorder on S.

Lemma 2.2. *Let μ, λ be right compatible preorders on S, $\theta_\mu, \theta_\lambda$ the right order congruences generated by μ, λ, respectively, K a right ideal of S, and $p, q \in S$. If $(pm)\theta_\lambda(qm)$ for every $[m]_{\theta_\mu} \in \overline{K}_{\theta_\mu}$, then $\mathcal{R}(K,\ \theta_\mu,\ \theta_\lambda,\ p) = \mathcal{R}(K,\ \theta_\mu,\ \theta_\lambda,\ q)$.*

Proof. Suppose that the given condition holds and $s\mathcal{R}(K,\ \theta_\mu,\ \theta_\lambda,\ p)t$. For every $a \in K(t,\ \theta_\mu)$, since $[sa]_{\theta_\mu}, [ta]_{\theta_\mu} \in \overline{K}_{\theta_\mu}$, it follows that $p(sa)\theta_\lambda q(sa)$ and $p(ta)\theta_\lambda q(ta)$ by hypothesis, and so

$$qsa = q(sa)\theta_\lambda p(sa) \underset{\lambda}{\leq} p(ta)\theta_\lambda q(ta) = qta.$$

This implies that $\mathcal{R}(K,\ \theta_\mu,\ \theta_\lambda,\ p) \subseteq \mathcal{R}(K,\ \theta_\mu,\ \theta_\lambda,\ q)$. Similarly we obtain that $\mathcal{R}(K,\ \theta_\mu,\ \theta_\lambda,\ q) \subseteq \mathcal{R}(K,\ \theta_\mu,\ \theta_\lambda,\ p)$. □

Lemma 2.3. *Let μ, λ be right compatible preorders on S, $\theta_\mu, \theta_\lambda$ the right order congruences generated by μ, λ, respectively, K a right ideal of S, and $p \in S$. Set $\rho = \mathcal{R}(K, \theta_\mu, \theta_\lambda, p)$. If $[m]_{\theta_\rho} \in \overline{K}_{\theta_\rho}$ then $[m]_{\theta_\mu} \in \overline{K}_{\theta_\mu}$ for all $[m]_{\theta_\rho} \in \overline{K}_{\theta_\rho}$.*

Proof. Let $[m]_{\theta_\rho} \in \overline{K}_{\theta_\rho}$. Then there exists $k \in K$ such that $[m]_{\theta_\rho} = [k]_{\theta_\rho}$. So $m \underset{\rho}{\leq} k$ and there exist $c_1, \cdots, c_s \in S$ such that

$$m \leq c_1 \rho c_2 \leq \cdots c_{s-1} \rho c_s \leq k.$$

Now K being strongly convex follows that $c_s \in K$, and so $[c_s]_{\theta_\mu} \in \overline{K}_{\theta_\mu}$, that is $1 \in K(c_s, \theta_\mu)$. Furthermore, since $c_{s-1} \rho c_s$, $1 \in K(c_{s-1}, \theta_\mu)$, we get that $[c_{s-1}]_{\theta_\mu} \in \overline{K}_{\theta_\mu}$. Consequently, the ρ-chain indicates that $[m]_{\theta_\mu} \in \overline{K}_{\theta_\mu}$. □

Now we are ready to give the main result of the paper. We characterize a completely strongly convex pomonoid S all of its cyclic S-posets are regular injective.

Theorem 2.4. *Let S be a completely strongly convex pomonoid. Then all cyclic S-posets are regular injective if and only if S has a left zero, and for any right ideal K of S, the right order congruences $\theta_\mu, \theta_\lambda$ generated by right compatible preorders μ and λ on S, respectively, and every S-poset homomorphism $f : \overline{K}_{\theta_\mu} \to S/\theta_\lambda$, there exists an element $q \in S$ such that*

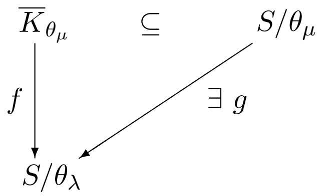
$$f([m]_{\theta_\mu}) = [q]_{\theta_\lambda} m,$$

for each $[m]_{\theta_\mu} \in \overline{K}_{\theta_\mu}$, and for $s, t \in S$,

$$s \; \mathcal{R}(K, \theta_\mu, \theta_\lambda, q) \; t \Rightarrow (qs) \underset{\lambda}{\leq} (qt).$$

Proof. Necessity. Suppose that all cyclic S-posets are regular injective. Then S_S is regular injective and so S_S has a zero by Lemma 1.6, which is a left zero of S.

Let K, θ_μ, θ_λ, f be as in the given conditions. Since S/θ_λ is regular injective, there exists an S-poset homomorphism $g:\ S/\theta_\mu \to S/\theta_\lambda$ such that the following diagram commutes.



Then $g([1]_{\theta_\mu}) = [p]_{\theta_\lambda}$ for some $p \in S$ and

$$g([m]_{\theta_\mu}) = g([1]_{\theta_\mu})m = [p]_{\theta_\lambda} m = f([m]_{\theta_\mu})$$

for all $[m]_{\theta_\mu} \in \overline{K}_{\theta_\mu}$. Set $\rho = \mathcal{R}(K, \theta_\mu, \theta_\lambda, p)$. Then ρ is a right compatible preorder on S by Lemma 2.1. Define $\alpha : \overline{K}_{\theta_\rho} \to S/\theta_\lambda$ by

$$\alpha([m]_{\theta_\rho}) = [p]_{\theta_\lambda} m$$

for any $[m]_{\theta_\rho} \in \overline{K}_{\theta_\rho}$. We claim that α is an S-poset homomorphism. $\square$

Firstly we show that α is well-defined. Suppose that $[m]_{\theta_\rho} = [n]_{\theta_\rho} \in \overline{K}_{\theta_\rho}$. Then $m\theta_\rho n$, and hence $m \underset{\rho}{\leq} n \underset{\rho}{\leq} m$. That is, there is a ρ-chain

$$m \leq a_1 \rho a_2 \leq \cdots \leq a_{l-1} \rho a_l \leq n \leq b_1 \rho b_2 \leq \cdots \leq b_{h-1} \rho b_h \leq m$$

from m to m, where $a_i, b_j \in S$. Therefore, by Lemma 1.1, we have

$$m \underset{\rho}{\leq} a_1 \underset{\rho}{\leq} \cdots \underset{\rho}{\leq} a_l \underset{\rho}{\leq} n \underset{\rho}{\leq} b_1 \underset{\rho}{\leq} \cdots \underset{\rho}{\leq} b_h \underset{\rho}{\leq} m.$$

This implies that

$$[m]_{\theta_\rho} \leq [a_1]_{\theta_\rho} \leq \cdots \leq [a_l]_{\theta_\rho} \leq [n]_{\theta_\rho} \leq [b_1]_{\theta_\rho} \leq \cdots \leq [b_h]_{\theta_\rho} \leq [m]_{\theta_\rho},$$

and then

$$[m]_{\theta_\rho} = [a_i]_{\theta_\rho} = [b_j]_{\theta_\rho} = [n]_{\theta_\rho}.$$

So $[a_i]_{\theta_\rho},\ [b_j]_{\theta_\rho} \in \overline{K}_{\theta_\rho}$. By Lemma 2.3, we have $[a_i]_{\theta_\mu} \in \overline{K}_{\theta_\mu}$ and then $1 \in K(a_i,\ \theta_\mu)$. Since $a_1 \rho a_2$, it follows that

$$(pa_1 1) = (pa_1) \underset{\lambda}{\leq} (pa_2) = (pa_2 1),$$

by the definition of ρ. Similarly, we have

$$(pa_3) \underset{\lambda}{\leq} (pa_4), \cdots, (pa_{l-1}) \underset{\lambda}{\leq} (pa_l), (pb_1) \underset{\lambda}{\leq} (pb_2), \cdots, (pb_{h-1}) \underset{\lambda}{\leq} (pb_h).$$

In addition,

$$m \leq a_1 \Rightarrow (pm) \leq (pa_1) \Rightarrow (pm) \underset{\lambda}{\leq} (pa_1).$$

Thus we obtain that

$$(pm) \underset{\lambda}{\leq} (pa_1) \underset{\lambda}{\leq} \cdots \underset{\lambda}{\leq} (pn) \underset{\lambda}{\leq} (pb_1) \underset{\lambda}{\leq} \cdots \underset{\lambda}{\leq} (pm).$$

It turns out that $(pm) \underset{\lambda}{\leq} (pn) \underset{\lambda}{\leq} (pm)$, that is,

$$[p]_{\theta_\lambda} m = [pm]_{\theta_\lambda} = [pn]_{\theta_\lambda} = [p]_{\theta_\lambda} n.$$

Consequently, α is well-defined.

Obviously, α preserves the S-action.

Now suppose that $[m]_{\theta_\rho} \leq [n]_{\theta_\rho}$. Similar to the proof of α being well-defined, there exist $a_1, a_2, \cdots, a_l \in S$ such that

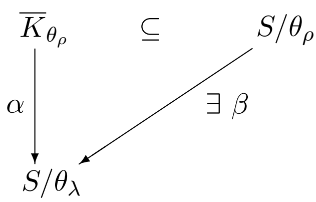
$$[m]_{\theta_\rho} \leq [a_1]_{\theta_\rho} \leq \cdots \leq [a_l]_{\theta_\rho} \leq [n]_{\theta_\rho},$$

and finally $(pm) \underset{\lambda}{\leq} (pn)$, which results in

$$[p]_{\theta_\lambda} m = [pm]_{\theta_\lambda} \leq [pn]_{\theta_\lambda} = [p]_{\theta_\lambda} n.$$

So α is order-preserving, and hence α is an S-poset homomorphism.

Since S/θ_λ is regular injective, there exists an S-poset homomorphism $\beta: S/\theta_\rho \to S/\theta_\lambda$ such that the following diagram commutes.



Then there exists an element $q \in S$ such that $\beta([1]_{\theta_\rho}) = [q]_{\theta_\lambda}$. We will show that $f([m]_{\theta_\mu}) = [q]_{\theta_\lambda} m$ for every $[m]_{\theta_\mu} \in \overline{K}_{\theta_\mu}$.

Assume that $[m]_{\theta_\mu} = [n]_{\theta_\mu} \in \overline{K}_{\theta_\mu}$. By the proof of Theorem 14 in [5], we obtain that $m \,\rho\, n$ and $n \,\rho\, m$. This implies that $m \underset{\rho}{\leq} n \underset{\rho}{\leq} m$ by Lemma 1.1. Hence, $[m]_{\theta_\rho} = [n]_{\theta_\rho}$. So for $[m]_{\theta_\mu} \in \overline{K}_{\theta_\mu}$, there exists $k \in K$ such that $[m]_{\theta_\mu} = [k]_{\theta_\mu}$. It follows that $[m]_{\theta_\rho} = [k]_{\theta_\rho}$, and then $[m]_{\theta_\rho} \in \overline{K}_{\theta_\rho}$ since $[k]_{\theta_\rho} \in \overline{K}_{\theta_\rho}$. Now for every $[m]_{\theta_\mu} \in \overline{K}_{\theta_\mu}$, we have
$f([m]_{\theta_\mu}) = [p]_{\theta_\lambda} m = \alpha([m]_{\theta_\rho}) = \beta([m]_{\theta_\rho}) = \beta([1]_{\theta_\rho} m) \qquad = \beta([1]_{\theta_\rho}) m = [q]_{\theta_\lambda} m.$

Now assume that $s \; \mathcal{R}(K,\, \theta_\mu,\, \theta_\lambda,\, q) \; t$. Since $[pm]_{\theta_\lambda} = [qm]_{\theta_\lambda}$, it follows that $\mathcal{R}(K, \theta_\mu, \theta_\lambda, q) = \mathcal{R}(K, \theta_\mu, \theta_\lambda, p) = \rho$ by Lemma 2.2. So

$$s \, \mathcal{R}(K,\, \theta_\mu,\, \theta_\lambda,\, q)\, t \Rightarrow s \underset{\mathcal{R}(K,\, \theta_\mu,\, \theta_\lambda,\, q)}{\leq} t \Rightarrow [s]_{\theta_{\mathcal{R}(K,\, \theta_\mu,\, \theta_\lambda,\, q)}} \leq [t]_{\theta_{\mathcal{R}(K,\, \theta_\mu,\, \theta_\lambda,\, q)}}.$$

Therefore,

$$\begin{aligned}
[q]_{\theta_\lambda} s &= \beta([1]_{\theta_\rho}) s \\
&= \beta([1]_{\theta_{\mathcal{R}(K,\ \theta_\mu,\ \theta_\lambda,\ q)}}) s \\
&= \beta([s]_{\theta_{\mathcal{R}(K,\ \theta_\mu,\ \theta_\lambda,\ q)}}) \\
&\leq \beta([t]_{\theta_{\mathcal{R}(K,\ \theta_\mu,\ \theta_\lambda,\ q)}}) \\
&= \beta([1]_{\theta_{\mathcal{R}(K,\ \theta_\mu,\ \theta_\lambda,\ q)}}) t \\
&= \beta([1]_{\theta_\rho}) t \\
&= [q]_{\theta_\lambda} t,
\end{aligned}$$

and hence $qs \underset{\lambda}{\leq} qt$ as desired.

Sufficiency. Assume that S has a left zero. Then every S-poset contains a zero element. Let S/θ_λ, S/θ_μ be cyclic S-posets, where θ_λ, θ_μ are right order congruences generated by right compatible preorders μ and λ on S, respectively. Note that for any S-subposet A of S/θ_μ, there exists a right ideal

$$K = \{a \in S \mid [a]_{\theta_\mu} \in A\}$$

of S such that $A = \overline{K}_{\theta_\mu}$. Let $f :\ \overline{K}_{\theta_\mu} \to S/\theta_\lambda$ be an S-poset homomorphism. Then by hypothesis, there exists $q \in S$ such that

$$f([m]_{\theta_\mu}) = [q]_{\theta_\lambda} m,$$

for every $[m]_{\theta_\mu} \in \overline{K}_{\theta_\mu}$, and

$$s\ \eta\ t \Rightarrow (qs) \underset{\lambda}{\leq} (qt),$$

where $\eta = \mathcal{R}(K,\ \theta_\mu,\ \theta_\lambda,\ q)$.

For each $[s]_{\theta_\mu} \in S/\theta_\mu$, define $g : S/\theta_\mu \to S/\theta_\lambda$ by

$$g([s]_{\theta_\mu}) = [q]_{\theta_\lambda} s.$$

Suppose that $[s]_{\theta_\mu} = [t]_{\theta_\mu}$. Again by the proof of Theorem 14 in [5], we have $s\eta t$ and $t\eta s$. So $qs \underset{\lambda}{\leq} qt \underset{\lambda}{\leq} qs$ by the hypothesis. Thus

$$g([s]_{\theta_\mu}) = [q]_{\theta_\lambda} s = [q]_{\theta_\lambda} t = g([t]_{\theta_\mu}),$$

which indicate that g is well-defined.

Next we show that g is order-preserving. Assume that $[s]_{\theta_\mu} \leq [t]_{\theta_\mu}$. Then $s \underset{\mu}{\leq} t$, and there exist $a_1, \cdots, a_n \in S$ such that

$$s \leq a_1 \mu a_2 \leq \cdots \leq a_{n-1} \mu a_n \leq t.$$

Now $a_1 \mu a_2$ implies that $a_1 \underset{\mu}{\leq} a_2$, i.e., $[a_1]_{\theta_\mu} \leq [a_2]_{\theta_\mu}$.

This results in $a_1 \eta a_2$ by the following reason. For any $x \in K(a_2, \theta_\mu)$, $[a_2 x]_{\theta_\mu} \in \overline{K}_{\theta_\mu}$ implies that $[a_1 x]_{\theta_\mu} \in \overline{K}_{\theta_\mu}$ since $\overline{K}_{\theta_\mu}$ is strongly convex. So $K(a_2, \theta_\mu) \subseteq K(a_1, \theta_\mu)$. Furthermore, for any $x \in K(a_2, \theta_\mu)$,

$$[q]_{\theta_\lambda} a_1 x = f([a_1 x]_{\theta_\mu}) \leq f([a_2 x]_{\theta_\mu}) = [q]_{\theta_\lambda} a_2 x$$

gives that $(q a_1 x) \underset{\lambda}{\leq} (q a_2 x)$. Therefore, $a_1 \eta a_2$ as required.

By the hypothesis, we have $(q a_1) \underset{\lambda}{\leq} (q a_2)$. Similarly, we obtain that

$$(q a_3) \underset{\lambda}{\leq} (q a_4), \ \cdots, \ (q a_{n-1}) \underset{\lambda}{\leq} (q a_n).$$

If $s \leq a_1$, then $(qs) \leq (q a_1)$, and so $(qs) \underset{\lambda}{\leq} (q a_1)$. By similar steps, we finally achieve that

$$(qs) \underset{\lambda}{\leq} (q a_1) \underset{\lambda}{\leq} (q a_2) \underset{\lambda}{\leq} (q a_3) \underset{\lambda}{\leq} \cdots \underset{\lambda}{\leq} (qt).$$

Therefore,

$$g([s]_{\theta_\mu}) = [qs]_{\theta_\lambda} \leq [qt]_{\theta_\lambda} = g[t]_{\theta_\mu}$$

result.

Consequently, S/θ_λ is regular injective.

Remark 2.5. Note that different from Theorem 14 in [5], where μ and λ are supposed to be right congruences on the monoid S, in this paper, we start from right compatible preorders, and result in that $\mathcal{R}(K,\ \theta_\mu,\ \theta_\lambda,\ q)$ is also a right compatible preorder, not necessarily a right congruence (see Lemma 2.1 and compare with Lemma 12 in [5]). This leads to conditions in Theorem which are different from those in the unordered case. Even if we specialize such that every S-poset is equipped with the discrete order as a partial order, μ, λ, $\mathcal{R}(K,\ \theta_\mu,\ \theta_\lambda,\ q)$ in Theorem are still not necessarily symmetric. In this sense, Theorem is a generalization of Theorem 14 in [5].

As an application, we present an example of a completely strongly convex pomonoid S all of its cyclic S-posets are regular injective.

Example 2.6. Let $S = \{0, 1, e, b\}$ be a semilattice with zero element 0, identity 1, and multiplication $eb = be = 0$. Let $(S, \leq)$ be the posemilattice equipped with the natural order. Consider the category $\mathscr{C}$, whose objects are S-posets equipped with the natural partial order, i.e., for an S-poset A, $a, b \in A$, $a \leq b \Leftrightarrow a = bs$ for some $s \in S$, and homomorphisms are S-poset homomorphisms. Then all S-posets in $\mathscr{C}$ are strongly convex.

For any S-poset homomorphism $\alpha :\ \overline{K}_{\theta_\mu} \to S/\theta_\lambda$, where K is an ideal of S, $\theta_\mu, \theta_\lambda$ are right order congruences generated by right compatible preorders μ and λ on S, respectively. Similar to [5] Example 15, we choose a suitable element q corresponding to α by discussing all non-trivial cases for the element e, and similarly for b.

Firstly, we have $\alpha([e]_{\theta_\mu}) = [0]_{\theta_\lambda}$ or $[e]_{\theta_\lambda}$.

Assume first that $[1]_{\theta_\mu} \notin \overline{K}_{\theta_\mu}$. If $[0]_{\theta_\mu} = [e]_{\theta_\mu} \in \overline{K}_{\theta_\mu}$ then $q = b$. If $[0]_{\theta_\mu} \neq [e]_{\theta_\mu} \in \overline{K}_{\theta_\mu}$ then $q = b$ if $[e]_{\theta_\lambda} = [0]_{\theta_\lambda}$, otherwise $q = 1$.

If $[1]_{\theta_\mu} \in \overline{K}_{\theta_\mu}$ then $q = 1$ if $\alpha([1]_{\theta_\mu}) = [1]_{\theta_\lambda}$, or $q = e$ if $\alpha([1]_{\theta_\mu}) = [e]_{\theta_\lambda}$.

Suppose $s\ \mathcal{R}(K, \theta_\mu, \theta_\lambda, q)\ t$, s, $t \in S$. It is easy to see that

$$K(0, \theta_\mu) = S,$$

$$K(e, \theta_\mu) = \begin{cases} \{0, b\} & \text{if } e \notin K, \\ S & \text{if } e \in K, \end{cases}$$

$$K(b, \theta_\mu) = \begin{cases} \{0, e\} & \text{if } b \notin K, \\ S & \text{if } b \in K, \end{cases}$$

$$K(1, \theta_\mu) = \begin{cases} S & \text{if } 1 \in K, \\ \{0,\ e,\ b\} & \text{if } e,\ b \in K, \\ \{0,\ e\} & \text{if } e \in K, \\ \{0,\ b\} & \text{if } b \in K, \\ \{0\} & \text{otherwise.} \end{cases}$$

For example, if $e \notin K$ then $K = \{0\}$ or $\{0, b\}$. Thus either $\overline{K}_{\theta_\mu} = \{[0]_{\theta_\mu}\}$ or $\overline{K}_{\theta_\mu} = \{[0]_{\theta_\mu}, [b]_{\theta_\mu}\}$. In both cases we have $K(e, \theta_\mu) = \{0, b\}$.

Next we give the proof how the conditions of Theorem are satisfied for $q = b$. What we need to show is that if $s, t \in S$ fulfilling $s\ \mathcal{R}(K, \theta_\mu, \theta_\lambda, q)\ t$ then one has $(qs) = (bs) \underset{\lambda}{\leq} (bt) = (qt)$. Let's prove under the following cases.

Case 1. If $s = 0$ or $s = e$ then we always have $bs = 0 \underset{\lambda}{\leq} (bt)$.

Case 2. Assume that $s = b$.

If $t = b$ or $t = 1$ then $bs = b \underset{\lambda}{\leq} b = (bt)$.

If $t = 0$ or $t = e$ then $b \in K(t, \theta_\mu)$. One has $b = (qsb) \underset{\lambda}{\leq} (qtb) = 0$. But this means $bs = b \underset{\lambda}{\leq} 0 = bt$.

Case 3. Assume that $s = 1$.

If $t = 1$ or $t = b$ then $bs = b \underset{\lambda}{\leq} b = (bt)$.

If $t = 0$ or $t = e \in K$ then $1 \in K(t, \theta_\mu)$. One has $b = (qs1) \underset{\lambda}{\leq} (qt1) = 0$, which implies that $bs = b \underset{\lambda}{\leq} 0 = bt$.

If $t = e \notin K$ then $b \in K(t, \theta_\mu)$, and so $b = (qsb) \underset{\lambda}{\leq} (qtb) = 0$. Again we get that $bs = b \underset{\lambda}{\leq} 0 = bt$.

Hence for $q = b$ we obtain that $s \; \mathcal{R}(K, \theta_\mu, \theta_\lambda, q) \; t \Rightarrow (qs) \underset{\lambda}{\leq} (qt)$ for all $s, t \in S$.

Similarly, by analyzing all the other possible cases of s and t in S, together with choosing suitable elements from $K(t, \theta_\mu)$, we obtain that $qs \underset{\lambda}{\leq} qt$ for $q = 1, e$. Therefore, we achieve that all cyclic S-posets are regular injective in the category $\mathscr{C}$ by the theorem in this work.

Acknowledgement. The authors appreciate the referees for the helpful comments and careful reading which lead to several improvements of the paper.

Bibliography

[1] S.L. Bloom, *Varieties of ordered algebras*, J. Comput. System Sci. 1976, 13, 200-212.

[2] S. Bulman-Fleming, V. Laan, *Lazard's theorem for S-posets*, Math. Nachr., 2005, 278(15), 1743-1755.

[3] M. Kilp, U. Knauer, A. Mikhalev, Monoids, Acts and Categories, with Applications to Wreath Products and Graphs; Walter de Gruyter: Berlin, New York, 2000.

[4] X. Zhang, *Regular injectivity of S-posets over Clifford pomonoids*, Southeast Asian Bull. Math., 2008, 32(5), 1007-1015.

[5] X. Zhang, U. Knauer, and Y.Q. Chen, *Classification of monoids by injectivities I. C-injectivity*, Semigroup Forum, 2008, 76(1), 169-176.

Xia Zhang, *School of Mathematical Sciences, South China Normal University, 510631 Guangzhou, China.*
Email: xiazhang@scnu.edu.cn

Wenling Zhang, *School of Mathematical Sciences, South China Normal University, 510631 Guangzhou, China.*
Email: z_wenling@126.com

Ulrich Knauer, *Institut fuer Mathematatik, Carl von Ossietzky University, D-26111 Oldenburg, Germany.*
Email: ulrich.knauer@uni-oldenburg.de